Sensor Motion for Optimal Estimation in Distributed Dynamic Environments

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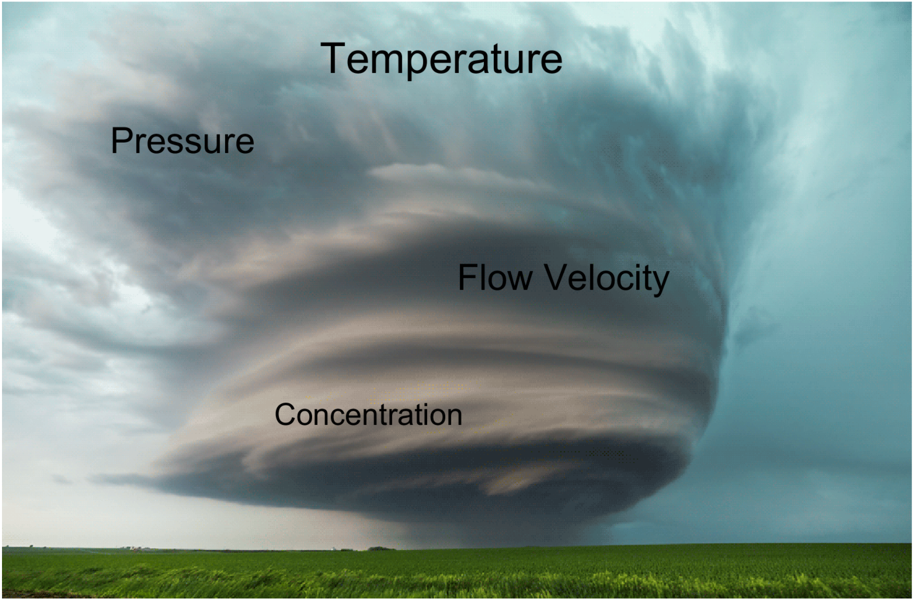

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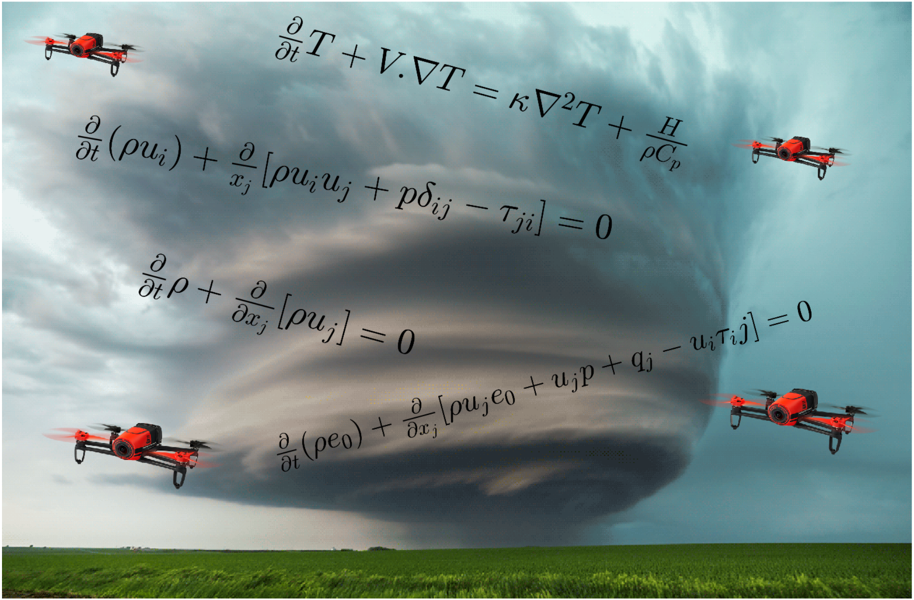
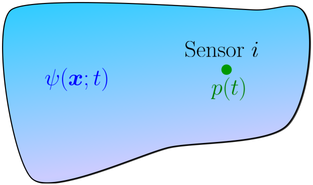
Dynamic Estimation: Incorporate the physical laws in the estimation process to reduce the number of sensors needed.



Overview

1. Part I: Estimation in Distributed Dynamic Environments
   - Measurement Schemes
   - Modeling Uncertain Dynamics: Linear PDE + Process Noise
   - Unknown Boundary Conditions
   - Case Study: Dynamic Acoustic Tomography
2. Part Two: Optimal Path Planning
   - Design Objective
   - Optimal Control Problem in Continuous Space-Time
   - Necessary Conditions of Optimality: State & Costate Equations
   - Case Study: Pointwise Sensor Path Design on 1D Heat Equation
   - Current & Future Work

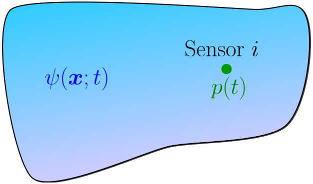
Pointwise Measurement Scheme



$\psi(\mathbf{x},t)$: unknown field to be estimated in space $\mathbf{x}$ and time t

$p(t)$: sensor position

Pointwise Measurement Scheme



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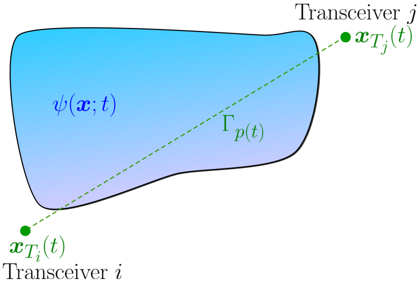
$p(t)$: sensor position

Measurement Equation: $m(t) = \mathcal{C}_{p(t)}\psi(t)$

$$\mathcal{C}_{p(t)}\psi := \psi(p(t); t)$$

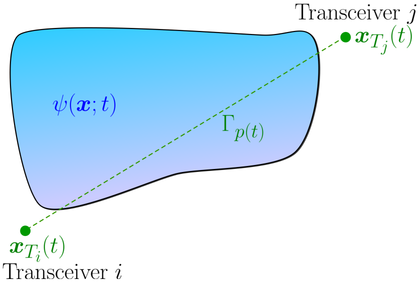
Pointwise Evaluation Operator

Tomographic Measurement Scheme: Line Integrals



$\Gamma_{p(t)}$: time varying line parametrized by $p(t)$

Tomographic Measurement Scheme: Line Integrals



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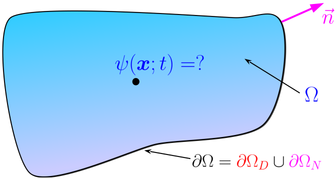
Measurement Equation: $m(t) = \mathcal{C}_{p(t)}\psi(t)$

$$\mathcal{C}_{p(t)}\psi := \int_{\Gamma_{p(t)}} \psi(\boldsymbol{x};t)d\boldsymbol{x}$$

Line Integral Operator

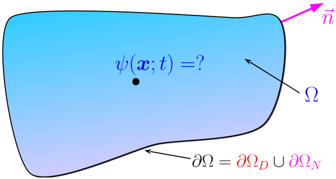


$$E\{w(\boldsymbol{x},t)w^*(\boldsymbol{\xi},\tau)\} = \mathcal{P}(\boldsymbol{x},\boldsymbol{\xi})\delta(t-\tau)$$

Spatial Correlation of PDE Model Uncertainty

Dynamics: $\frac{\partial}{\partial t}\psi(t) = \mathcal{A}\psi(t) + w(t);$ $\psi(0) = \psi_0$

Unknown Boundary Conditions as "Process Noise"



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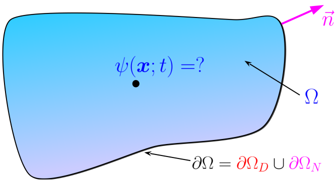
BC: $\psi(t)\Big|_{\partial\Omega_D} = \psi_D(t) \qquad \frac{\partial}{\partial \vec{n}}\psi(t)\Big|_{\partial\Omega_N} = \psi_N(t)$

↑ Dirichlet ↑ Neumann

Unknown Boundary Conditions as "Process Noise"



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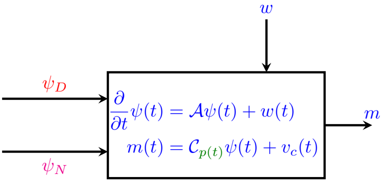

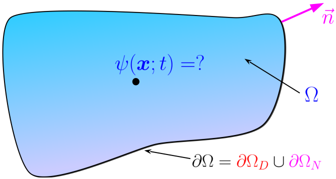

$$E\{w(\mathbf{x},t)w^*(\boldsymbol{\xi},\tau)\} = \mathcal{P}(\mathbf{x},\boldsymbol{\xi})\delta(t-\tau)$$
$$E\{w_D(\mathbf{x},t)w_D^*(\boldsymbol{\xi},\tau)\} = \mathcal{P}_D(\mathbf{x},\boldsymbol{\xi})\delta(t-\tau)$$
$$E\{w_N(\mathbf{x},t)w_N^*(\boldsymbol{\xi},\tau)\} = \mathcal{P}_N(\mathbf{x},\boldsymbol{\xi})\delta(t-\tau)$$

Dynamics: $\frac{\partial}{\partial t}\psi(t) = \mathcal{A}\psi(t) + w(t); \qquad \psi(0) = \psi_0$

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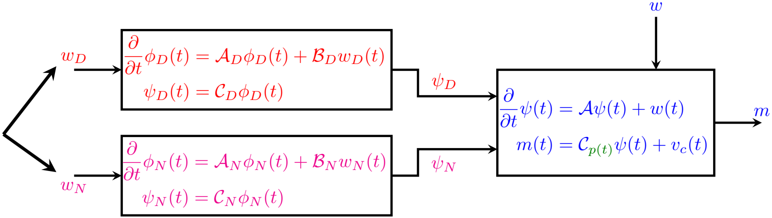
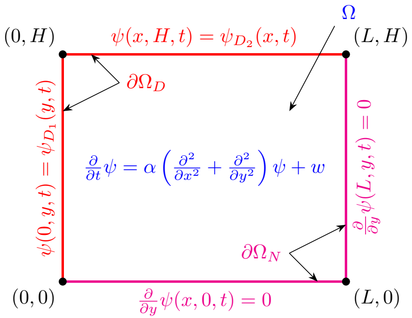
BC Process Noise



Case Study: Dynamic Acoustic Tomography



Unknown, Time-Varying Dirichlet BC
(Heated Walls)

Homogeneous Neumann BC
(Insulated Walls)

$$\psi(0, y, t) = 20 + 10\sin\left(\frac{2\pi}{24 \times 60}t\right)$$

$$\psi(x, H, t) = 30 - 10\sin\left(\frac{2\pi}{24 \times 60}t\right)$$

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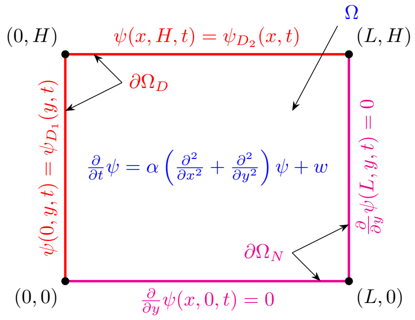
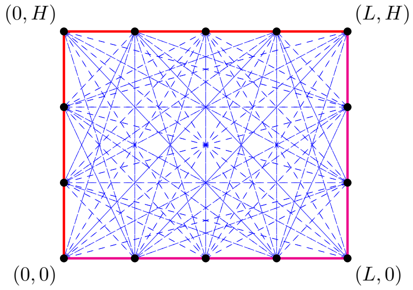
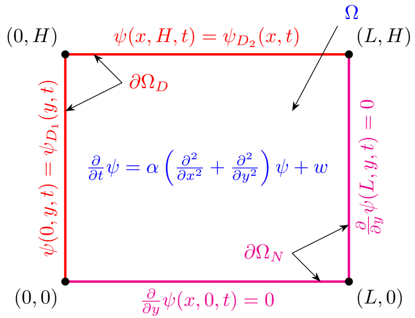
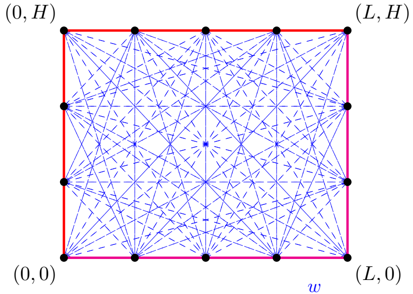


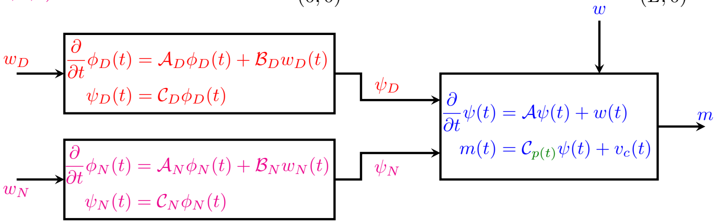


- Ultrasonic transceivers measure the Time of Flight of sound waves.
- Time of Flight depends on the line integral of the temperature field.

Case Study: Dynamic Acoustic Tomography







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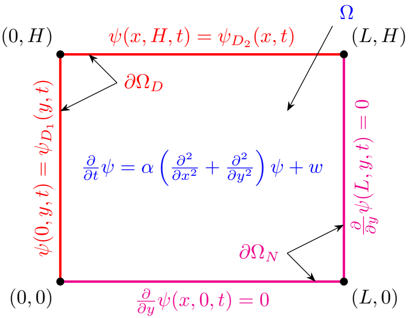

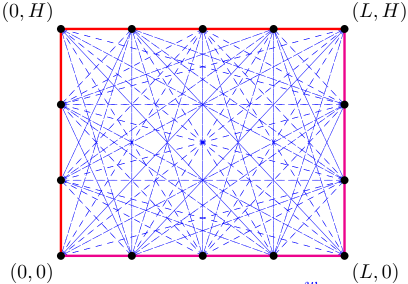

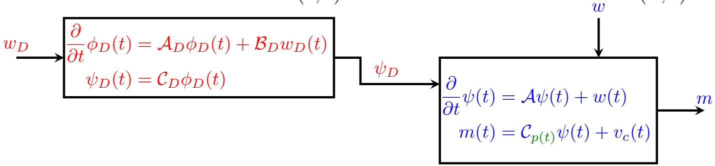

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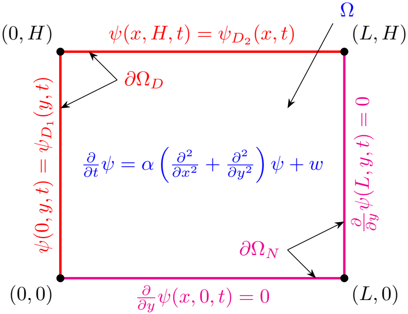
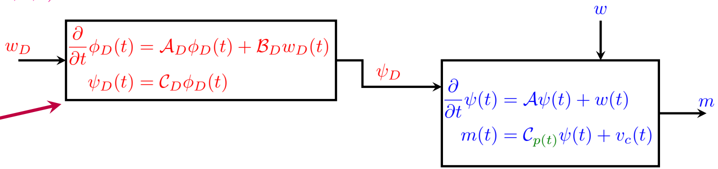


Design Parameters of the Uncertain BC:

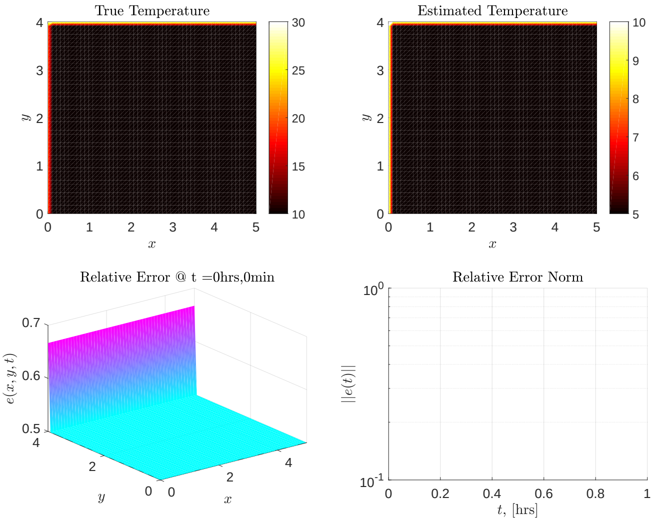
- ω_c: Time scale
- a_i: Magnitude
- σ_i: Correlation length

Low Pass Filter: ω_c



$$\mathcal{P}_D=\begin{bmatrix}\mathcal{P}_{D_1} & 0\\ 0 & \mathcal{P}_{D_2}\end{bmatrix} \qquad \mathcal{P}_{D_i}(x,\xi)=a_i e^{-\frac{(x-\xi)^2}{\sigma_i^2}}; \quad i=1,2$$

Perfect Knowledge of the Diffusion Coefficient



Perfect Knowledge of the Diffusion Coefficient

Severe Perturbation in the Diffusion Coefficient

Plan

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Design Objective

$\psi(x,t)$: augmented state space variable
$w(x,t)$: augmented process noise
$v(t)$: measurement noise

Augmented Dynamics:

$$\begin{cases} \frac{\partial}{\partial t}\psi(x,t) = \mathcal{A}\psi(\mathbf{x},t) + w(t); \qquad \psi(\mathbf{x},0) = \psi_0(\mathbf{x}) \\ m(t) = \mathcal{C}_{p(t)}\psi(\mathbf{x},t) + v(t) \end{cases}$$

$\longrightarrow$ **Goal**: Design the path $p(t)$ to minimize the estimation error in some sense.

Optimal Control Problem in Continuous Space-Time

$\hat{\psi}(\mathbf{x}, t)$	$\longrightarrow$	**Optimal** State Estimate
$e(\mathbf{x}, t) := \psi(\mathbf{x}, t) - \hat{\psi}(\mathbf{x}, t)$	$\longrightarrow$	Estimation Error
$E\{e(\mathbf{x}, t)e^*(\boldsymbol{\xi}, \tau)\} := \mathcal{X}(\boldsymbol{x}, \boldsymbol{\xi}; t)\delta(t - \tau)$	$\longrightarrow$	Estimation Error Covariance

Optimal Control Problem in Continuous Space-Time

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$$\implies \text{trace}(\mathcal{X}(t)) = E\{\int e^*(\boldsymbol{\xi}, t)e(\boldsymbol{\xi}, t)d\boldsymbol{\xi}\} = E\{||e(t)||_{L_2}^2\}$$

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Objective:

- Design$\{$p(t)$\}$ to minimize tr$(\mathcal{X})$
- Add some penalty on the sensors' mobility

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Objective:
- Design$\{p(t)\}$ to minimize tr$(\mathcal{X})$
- Add some penalty on the sensors' mobility

$$\min_{\{z(t);\mathcal{X}(t)\}} \int_0^{t_f} \left(tr(\mathcal{X}(t)) + \frac{1}{2}z(t)^T Q_s z(t) + \frac{1}{2}u(t)^T R_s u(t) \right) dt$$

Dynamics of Error Covariance:

$$\frac{\partial}{\partial t}\mathcal{X} = \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^* + \mathcal{Q} - \mathcal{X}\mathcal{C}_p^* R^{-1}\mathcal{C}_p\mathcal{X}; \quad \mathcal{X}(0) = \mathcal{X}_0$$

Sensor Dynamics:

$$\frac{d}{dt}z = Fz + Gu; \quad z(0) = z_0$$

$$p = Hz$$

Deterministic Optimal Control Problem

Necessary Conditions of Optimality: States & Costates

- **Covariance State & Costate:** $\mathcal{X} \longleftrightarrow \mathcal{Y}$

$$\frac{\partial}{\partial t}\mathcal{X} = \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^* + \mathcal{Q} - \mathcal{X}\mathcal{C}_p^* R^{-1}\mathcal{C}_p\mathcal{X}; \qquad \mathcal{X}(0) = 0$$

$$-\frac{\partial}{\partial t}\mathcal{Y} = (\mathcal{A} - \mathcal{L}_p\mathcal{C}_p)^*\,\mathcal{Y} + \mathcal{Y}\,(\mathcal{A} - \mathcal{L}_p\mathcal{C}_p) + \mathcal{I}; \qquad \mathcal{Y}(t_f) = 0$$

where $\mathcal{L}_p := \mathcal{X}\mathcal{C}_p R^{-1}$ is the Kalman Gain.

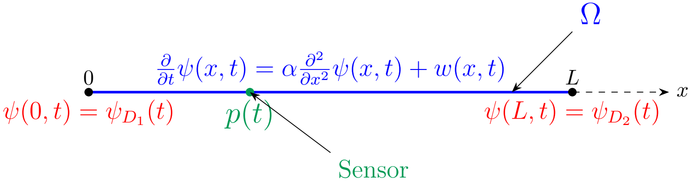
- **Sensor State & Costate Equation:** $z \longleftrightarrow \lambda$

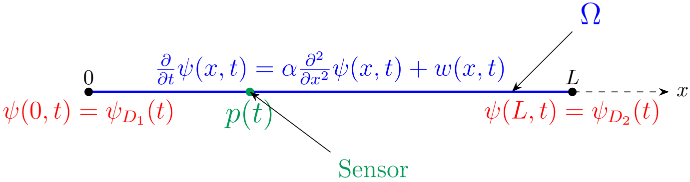
$$\frac{d}{dt}z = Fz + Gu; \qquad (u = -R_s^{-1}G^T\lambda); \qquad z(0) = 0$$

$$-\frac{d}{dt}\lambda = F^T\lambda + Q_s z - H^T \mathrm{tr}\left(\mathcal{X}\mathcal{W}_p\mathcal{X}\mathcal{Y}\right); \qquad \lambda(t_f) = 0$$

where $\mathcal{W}_p := \frac{\partial}{\partial p}(\mathcal{C}_p^* R^{-1}\mathcal{C}_p)$ and $p = Hz$

Case Study: Sensor Path Design on 1D Heat Equation





Approximate Optimal Sensor Path: [1]

- Discretize time: $t_k := k\Delta$
- Given $p(t_{k-1})$ and $\mathcal{X}(t_{k-1})$, compute $p(t_k)$ that minimizes:

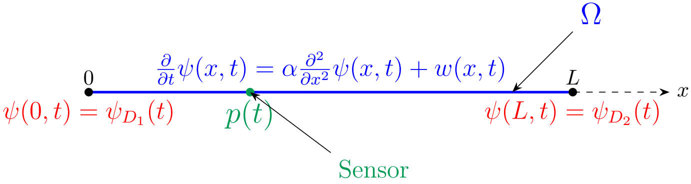
$$\mathcal{J}(p(t_k)) = tr(\mathcal{X}(t_k)) + \frac{\mu}{2}\left(\frac{p(t_k) - p(t_{k-1})}{\Delta}\right)^2$$

Estimation Error

Mobility Penalty

[1] Related to Choi, H. L., & How, J. P. (2010). Continuous trajectory planning of mobile sensors for informative forecasting.

cut off frequency: f_n BC Process Noise Variance: $\mathcal{P}_{D_1}$ and $\mathcal{P}_{D_2}$



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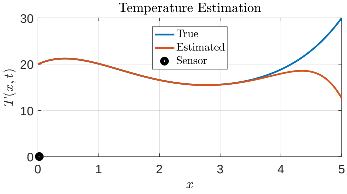
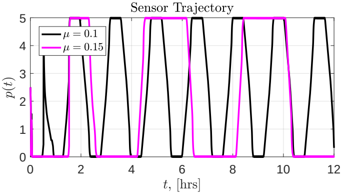
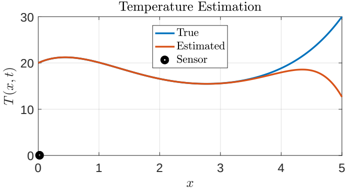
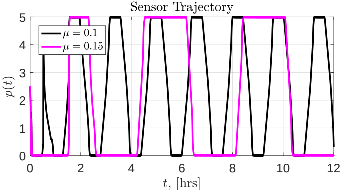
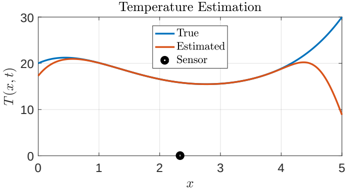


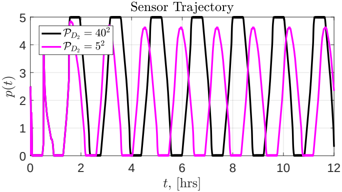


Case Study: Sensor Path Design on 1D Heat Equation

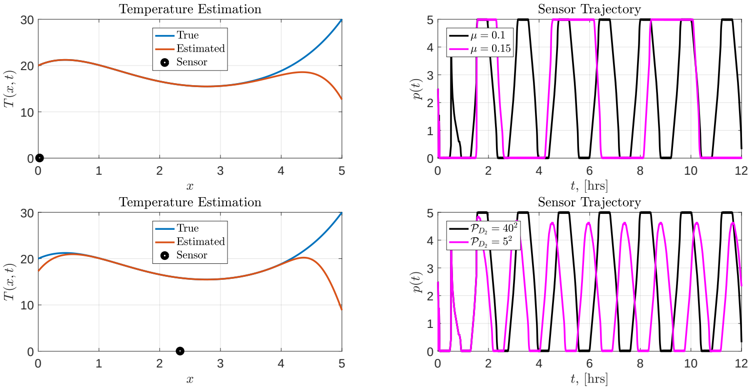








Case Study: Sensor Path Design on 1D Heat Equation



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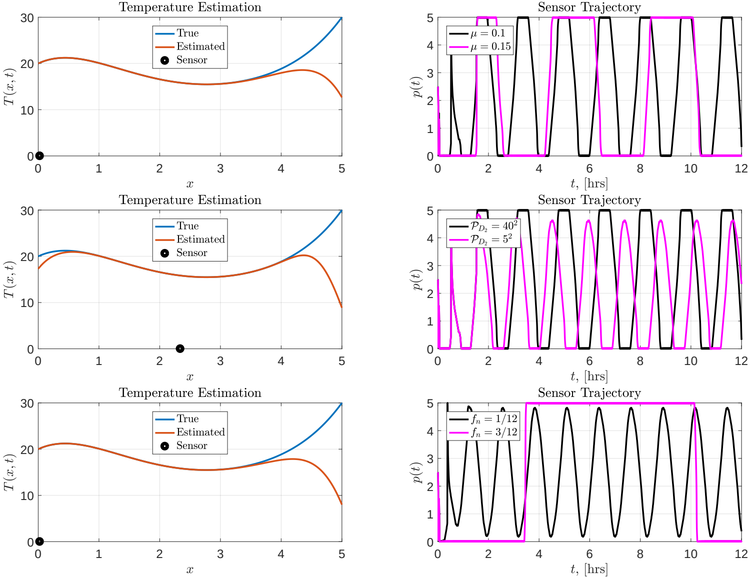

Current & Future Work

- Understand the structure of the State/Costate Differential Equations
- Devise efficient numerical methods
- Cooperative multi-agent path planning
- Further generalizations to nonlinear distributed dynamical systems
- Application to Navier-Stokes Equations

Questions?

Kalman Filter Set up

$\longrightarrow$ **Key**: Absorb the modeled dynamics of the unknown boundary conditions.

$$\psi_c := \begin{bmatrix} \psi \\ \phi_D \\ \phi_N \end{bmatrix} \qquad w_c := \begin{bmatrix} w \\ w_D \\ w_N \end{bmatrix}$$

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$$\psi_c := \begin{bmatrix} \psi \\ \phi_D \\ \phi_N \end{bmatrix} \qquad w_c := \begin{bmatrix} w \\ w_D \\ w_N \end{bmatrix}$$

$$\begin{cases} \dot{\psi}_c(t) = \mathcal{A}_c \psi_c(t) + w_c(t); & \psi_c(0) = \psi_{c0} \\ m(t) = \begin{bmatrix} \mathcal{C}_{p(t)} & 0 \end{bmatrix} \psi_c(t) + v_c(t) \end{cases}$$

$$E\{w_c(\boldsymbol{x}, t) w_c^*(\boldsymbol{\chi}, \tau)\} = \mathcal{Q}_c(\boldsymbol{x}, \boldsymbol{\chi}) \delta(t - \tau) \quad \longrightarrow \text{Process Noise}$$

$$E\{v_c(t) v_c^T(\tau)\} = R_c \delta(t - \tau) \quad \longrightarrow \text{Measurement Noise}$$

$$\mathcal{A}_c := \begin{bmatrix} \mathcal{A} & 0 & 0 \\ 0 & \mathcal{A}_D & 0 \\ 0 & 0 & \mathcal{A}_N \end{bmatrix} \qquad \mathcal{Q}_c := \begin{bmatrix} \mathcal{P}_w & 0 & 0 \\ 0 & \mathcal{B}_D \mathcal{P}_D \mathcal{B}_D^* & 0 \\ 0 & 0 & \mathcal{B}_N \mathcal{P}_N \mathcal{B}_N^* \end{bmatrix}$$