A Block Diagram Approach to Stochastic Calculus with Application to Multiplicative Uncertainty Analysis

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CDC 2018, Miami

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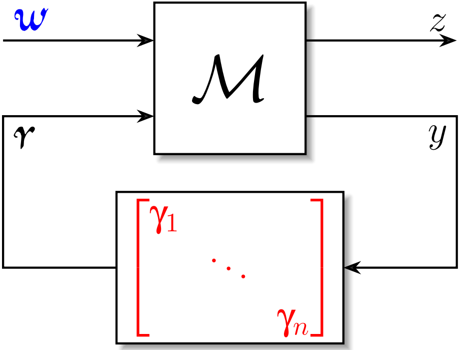
Overview

- Problem Statement: Structured Stochastic Uncertainty
- Stochastic Block Diagrams
- Block Diagram Conversion Between Stochastic Interpretations
- Loop Gain Operator
- Conditions for Mean-Square Stability
- Final Remarks & Conclusion

Structured Stochastic Uncertainty: Input/Output Approach



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$$\mathbb{E}\left[\gamma(t)\gamma^*(\tau)\right] := \mathbf{\Gamma}\delta(t-\tau)$$

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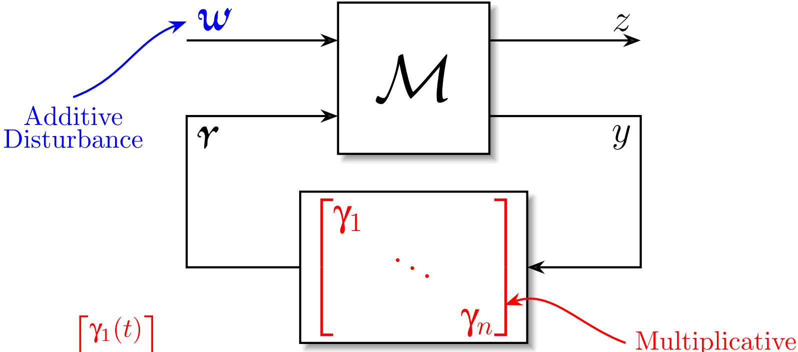

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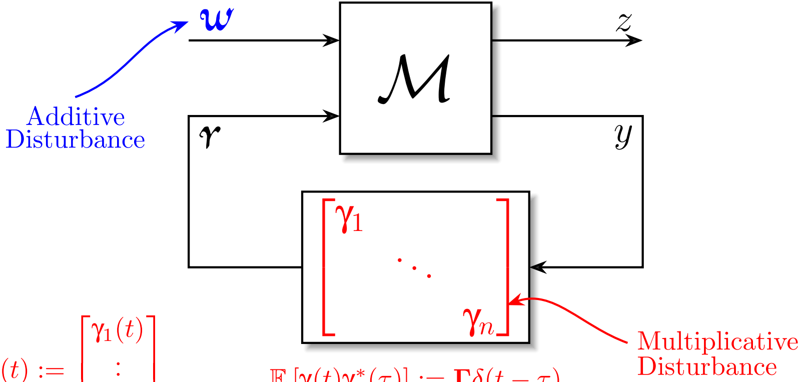

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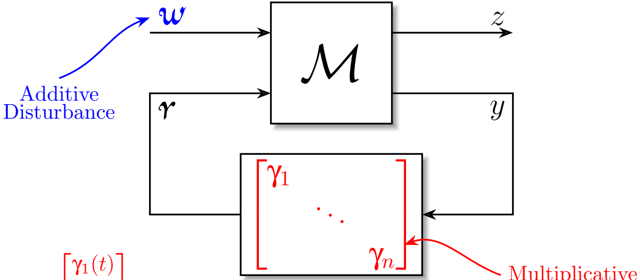
$$\begin{bmatrix} z \\ y \end{bmatrix} = \mathcal{M}\begin{bmatrix} w \\ r \end{bmatrix} \iff \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \int_0^t M(t-\tau)\begin{bmatrix} w(\tau) \\ r(\tau) \end{bmatrix} d\tau$$

$$r(t) = \mathsf{Diag}\Big(\gamma(t)\Big)y(t).$$

Goal: What are the conditions of MSS?



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Mean-Square Stability & Structured Stochastic Uncertainty

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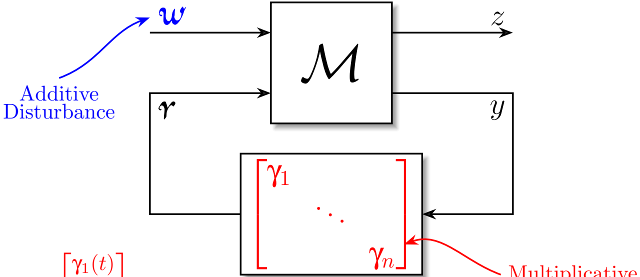

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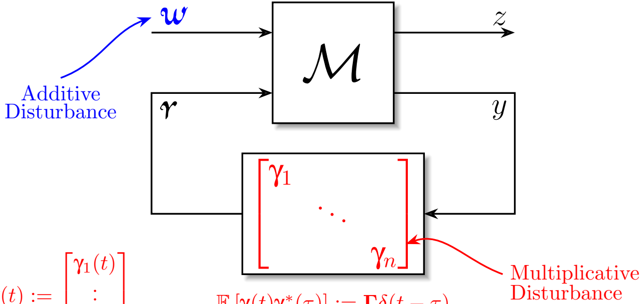

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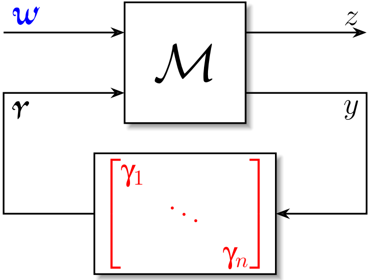
Stochastic Block Diagrams



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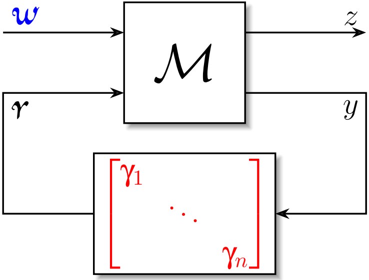

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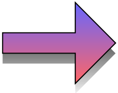

$$\gamma := \frac{d\gamma}{dt}$$

$$r := \frac{dr}{dt}$$

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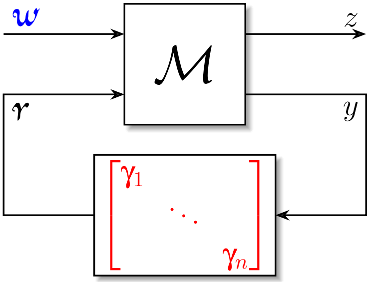

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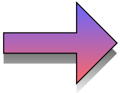
White Process Representation

$$w := \frac{dw}{dt}$$



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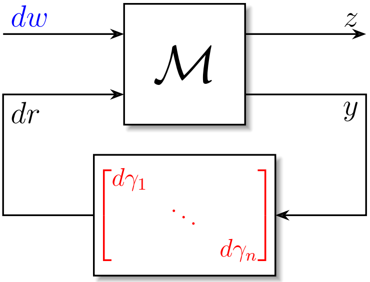

$$\mathbb{E}\left[d\gamma(t)d\gamma^*(t)\right] = \mathbf{\Gamma}dt$$
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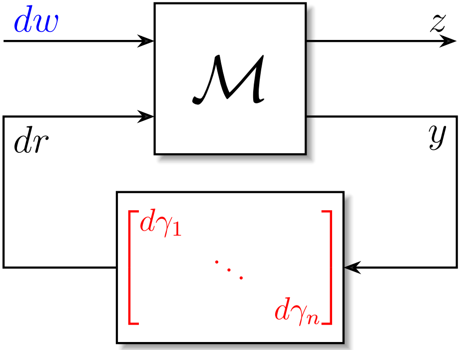
Wiener Process Representation

$$\begin{bmatrix} z \\ y \end{bmatrix} = \mathcal{M}\begin{bmatrix} dw \\ dr \end{bmatrix} \Longleftrightarrow \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \int_0^t M(t-\tau)\begin{bmatrix} dw(\tau) \\ dr(\tau) \end{bmatrix}$$
$$dr(t) = \text{Diag}\big(d\gamma(t)\big)y(t).$$

Stochastic Interpretations: Itō & Stratonovich



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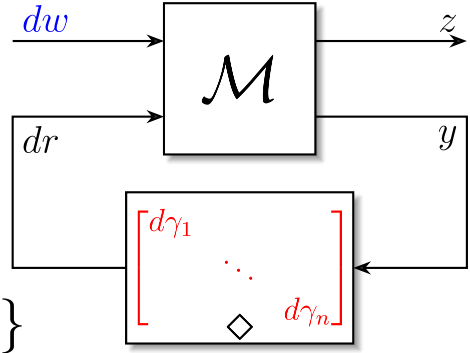

$$\diamond \in \{\diamond_I, \diamond_S\}$$

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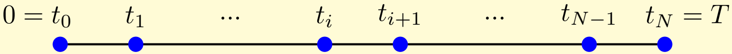
Recall: Stochastic Integrals



- **Deterministic:**

$$\int_0^T v(t)dt := \lim_{N\to\infty} \sum_{k=0}^{N-1} v(\bar{t}_i)\Big(t_{i+1} - t_i\Big); \quad \forall \bar{t}_i \in [t_i, t_{i+1}]$$

- **Itō:**

$$\int_0^T v(t) \diamond_I dw(t) := \lim_{N\to\infty} \sum_{i=0}^{N-1} v(t_i)\Big(w(t_{i+1}) - w(t_i)\Big)$$

- **Stratonovich:**

$$\int_0^T v(t) \diamond_S dw(t) := \lim_{N\to\infty} \sum_{i=0}^{N-1} v\left(\frac{t_i + t_{i+1}}{2}\right)\Big(w(t_{i+1}) - w(t_i)\Big)$$

Recall: Stochastic Integrals

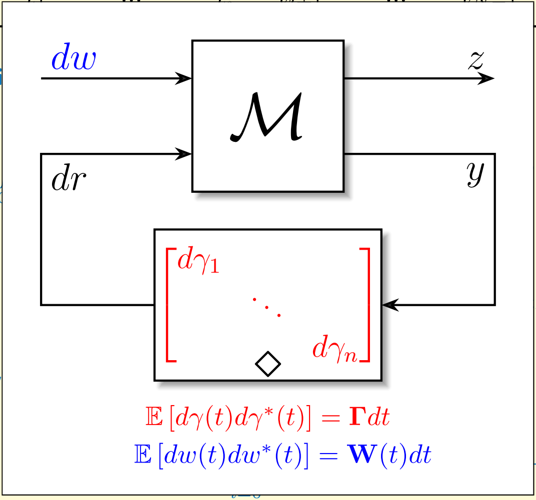
$0 = t_0$ ... $t_N = T$

- **Determini**

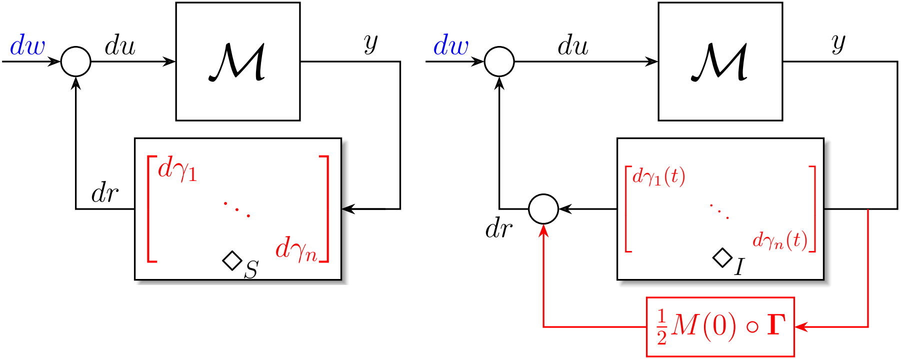
$t_{i+1}]$

- **Itō:**

- **Stratonov**



Stratonovich to Ito Conversion

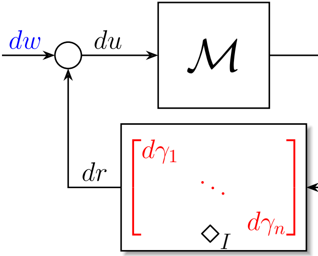


$\mathbb{E}\left[d\gamma(t)d\gamma^*(t)\right] := \mathbf{\Gamma}dt;$ $\qquad y(t) = \int_0^t M(t-\tau)du(\tau);$

" $\circ$ "is the Hadamard (element-by-element) product

The two stochastic block diagrams are "equivalent in the mean-square sense".

Loop Gain Operator & Mean-Square Stability



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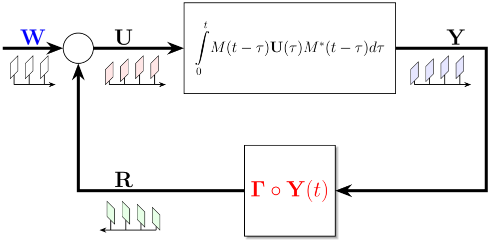
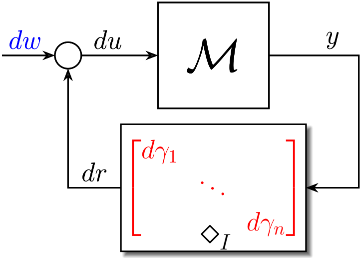


$\mathbb{E}\left[du(t)du^*(t)\right] = \mathbf{U}(t)dt;$ $\mathbb{E}\left[y(t)y^*(t)\right] = \mathbf{Y}(t);$

$\mathbb{E}\left[dr(t)dr^*(t)\right] = \mathbf{R}(t)dt;$ "$\circ$": Hadamard Product;

Deterministic Covariance Block Diagram

Loop Gain Operator & Mean-Square Stability



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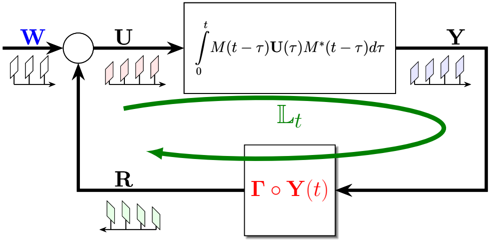

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$$\mathbb{L}_t(\mathbf{U}) := \mathbf{\Gamma} \circ \left(\int_0^t M(t-\tau)\mathbf{U}(\tau)M^*(t-\tau)d\tau \right), \qquad \mathbb{L} := \lim_{t\to\infty} \mathbb{L}_t$$

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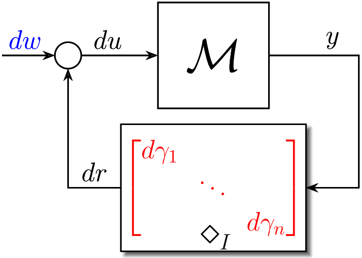

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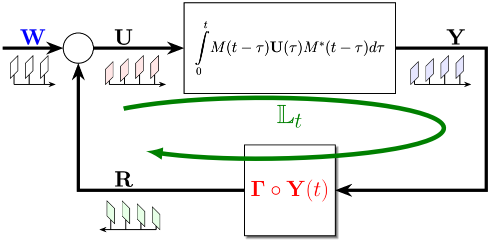

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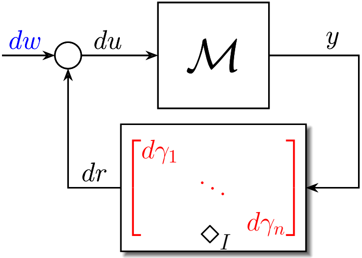
Necessary & Sufficient Conditions of Mean-Square Stability:

- Forward Block is Stable (Finite H^2-norm)
- Spectral Radius of $\mathbb{L}$ is strictly less than 1, $\rho(\mathbb{L}) < 1$

Loop Gain Operator & Mean-Square Stability



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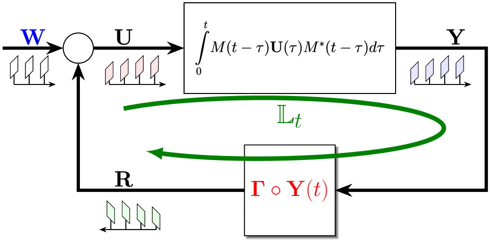

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Two important quantities related to $\mathbb{L}$:

- Spectral Radius: $\rho(\mathbb{L})$
- Worst-Case Covariance: $\mathbb{L}(\hat{\mathbf{U}}) = \rho(\mathbb{L})\hat{\mathbf{U}}$ (Perron-Frobenius "Eigen-matrix")

Proofs & An Application

- **Discrete-Time Setting**: Bamieh, B., & Filo, M. (2018). An Input-Output Approach to Structured Stochastic Uncertainty. arXiv preprint arXiv:1806.07473.
- **Continuous-Time Setting**: Filo, M., & Bamieh, B. (2018). An Input-Output Approach to Structured Stochastic Uncertainty in Continuous Time. arXiv preprint arXiv:1806.09091.

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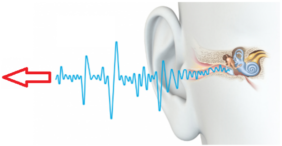
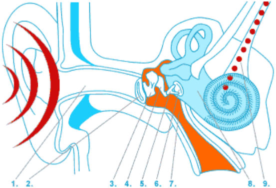
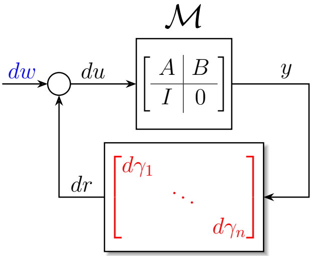
- **Application:** Stochastic Instabilities in the inner ear!





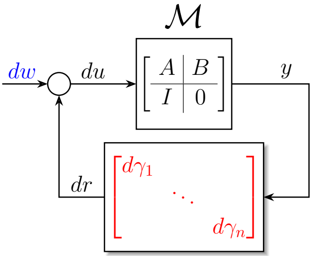
Filo, M., & Bamieh, B. (2017, December). Investigating cochlear instabilities using structured stochastic uncertainty. In Decision and Control (CDC), 2017 IEEE 56th Annual Conference on (pp. 1634-1640). IEEE.

Concluding Remarks & Future Work



SDE: $dy(t) = Ay(t)dt + B\text{Diag}\Big(d\gamma(t)\Big)y(t) + Bdw(t)$

Concluding Remarks & Future Work



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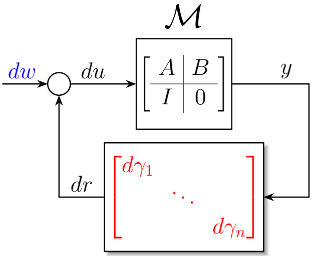
Extends and unifies the analysis for systems $\mathcal{M}$:

- State space realizations
- Infinite dimensional systems with finite number of multiplicative disturbances
- Systems with delays

Concluding Remarks & Future Work



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Extends and unifies the analysis for systems $\mathcal{M}$:

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Future Direction: Extend the analysis for

- *Colored* disturbances
- Spatially distributed disturbances with symmetries.