Topics in Stochastic Stability, Optimal Control & Estimation Theory

Maurice Filo

PhD Dissertation Defense

https://engineering.ucsb.edu/~filo/

University of California, Santa Barbara

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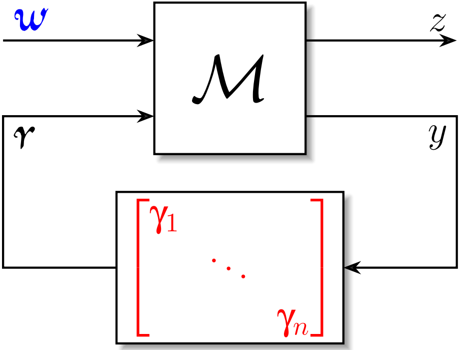
Committee: Igor Mezic
Joao Hespanha
Francesco Bullo

June 8, 2018

Overview

Plan

Mean-Square Stability & Structured Stochastic Uncertainty



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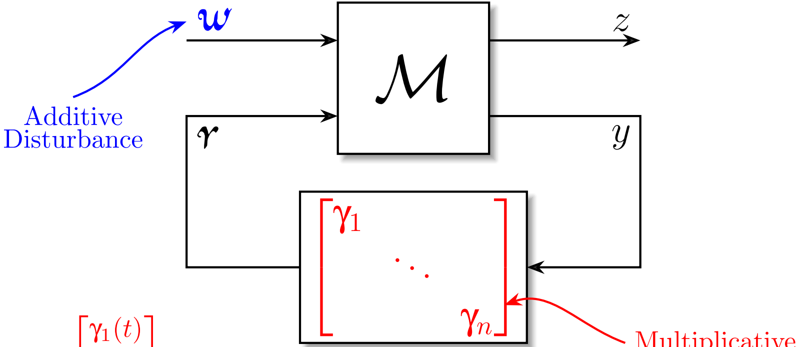

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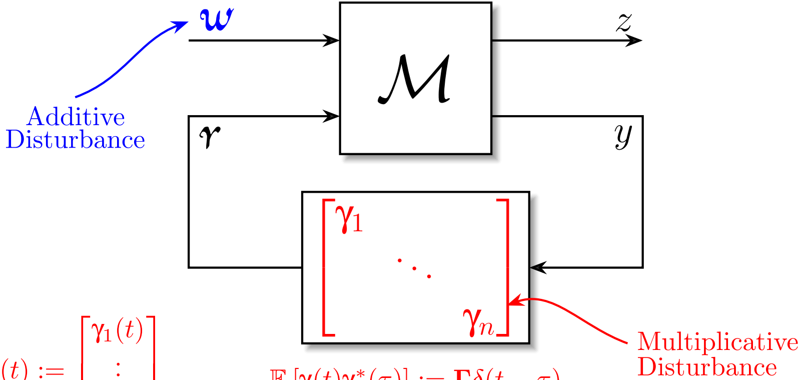

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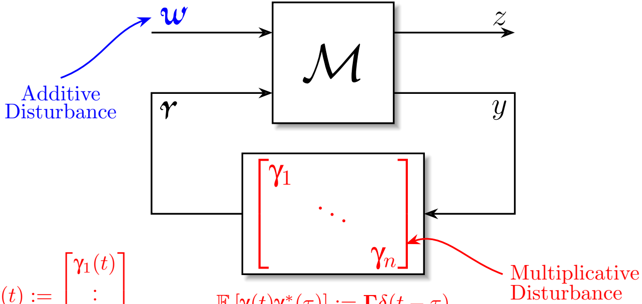
$$r(t) = \mathsf{Diag}\Big(\gamma(t)\Big)y(t).$$

Goal: What are the conditions of MSS?



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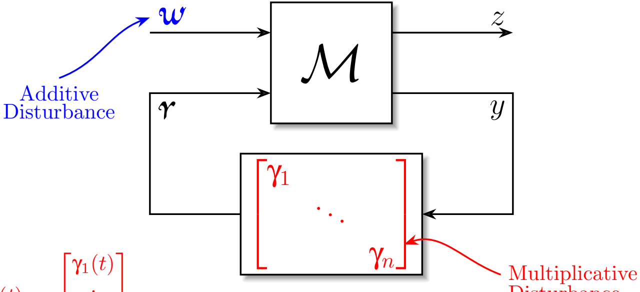

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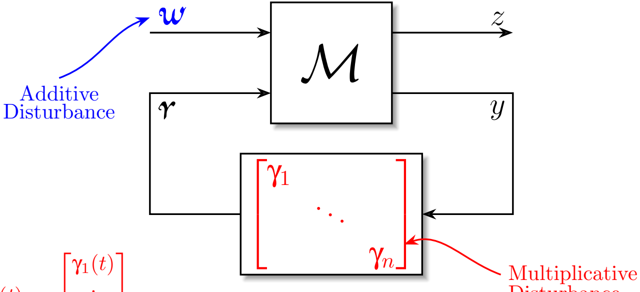

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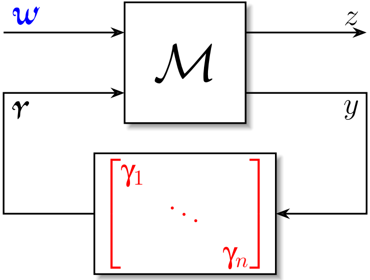
Stochastic Block Diagrams



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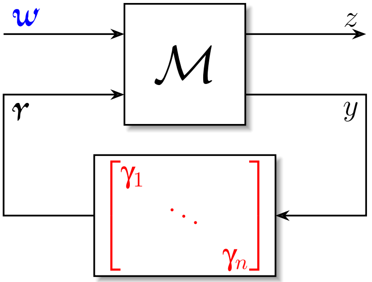

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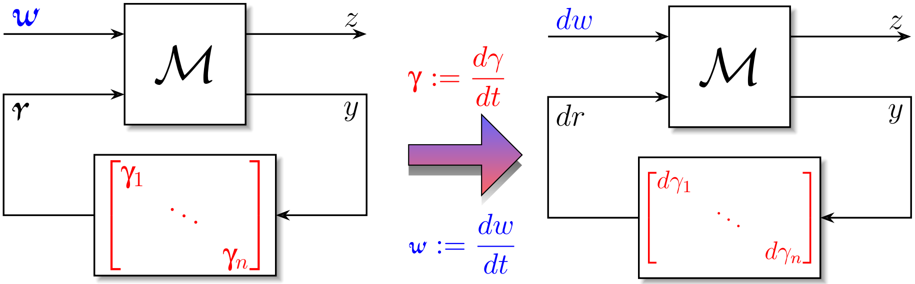
$$\boldsymbol{\gamma} := \frac{d\gamma}{dt}$$

$$\boldsymbol{w} := \frac{dw}{dt}$$



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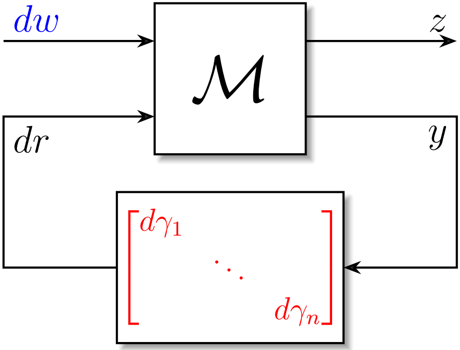
White Process Representation

$$\mathbb{E}\left[d\gamma(t)d\gamma^*(t)\right] = \mathbf{\Gamma}dt$$
$$\mathbb{E}\left[dw(t)dw^*(t)\right] = \mathbf{W}(t)dt$$

Wiener Process Representation

$$\begin{bmatrix} z \\ y \end{bmatrix} = \mathcal{M}\begin{bmatrix} dw \\ dr \end{bmatrix} \iff \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \int_0^t M(t-\tau)\begin{bmatrix} dw(\tau) \\ dr(\tau) \end{bmatrix}$$
$$dr(t) = \mathrm{Diag}\big(d\boldsymbol{\gamma}(t)\big)y(t).$$

Stochastic Interpretations: Itō & Stratonovich



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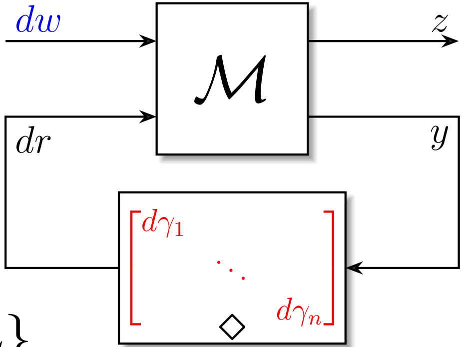

$$\diamond \in \{\diamond_I, \diamond_S\}$$

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$$\begin{bmatrix} z \\ y \end{bmatrix} = \mathcal{M} \begin{bmatrix} dw \\ dr \end{bmatrix} \Longleftrightarrow \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} = \int_0^t M(t-\tau) \diamond \begin{bmatrix} dw(\tau) \\ dr(\tau) \end{bmatrix}$$

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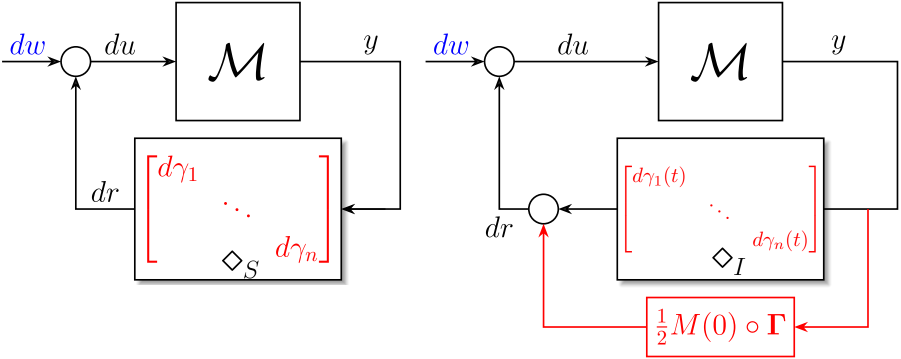
Stratonovich to Ito Conversion

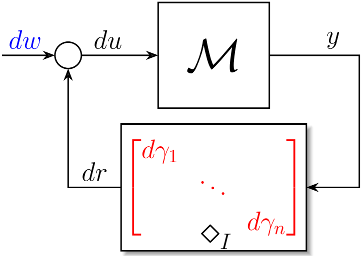


$$\mathbb{E}\left[d\gamma(t)d\gamma^*(t)\right] := \mathbf{\Gamma}dt; \qquad y(t) = \int_0^t M(t-\tau)du(\tau);$$

"$\circ$" is the Hadamard (element-by-element) product

The two stochastic block diagrams are equivalent in the mean-square sense!

Loop Gain Operator & Mean-Square Stability



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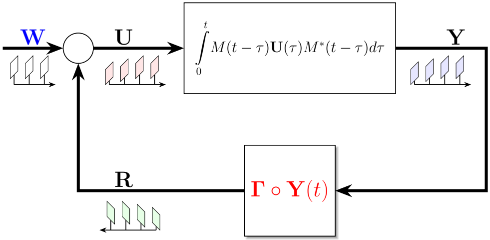
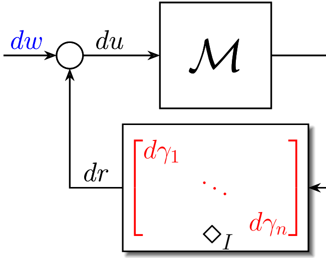


$\mathbb{E}\left[du(t)du^*(t)\right] = \mathbf{U}(t)dt;$ $\quad \mathbb{E}\left[y(t)y^*(t)\right] = \mathbf{Y}(t);$

$\mathbb{E}\left[dr(t)dr^*(t)\right] = \mathbf{R}(t)dt;$ $\quad$ "$\circ$": Hadamard Product;

Deterministic Covariance Block Diagram



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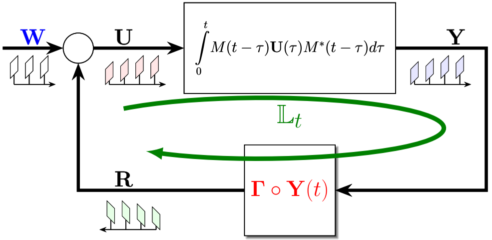

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$$\mathbb{L}_t(\mathbf{U}) := \mathbf{\Gamma} \circ \left(\int_0^t M(t-\tau)\mathbf{U}(\tau)M^*(t-\tau)d\tau \right), \qquad \mathbb{L} := \lim_{t\to\infty} \mathbb{L}_t$$

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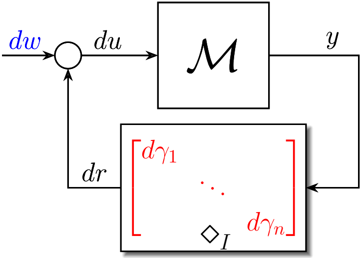

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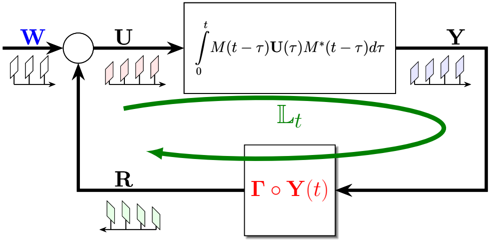

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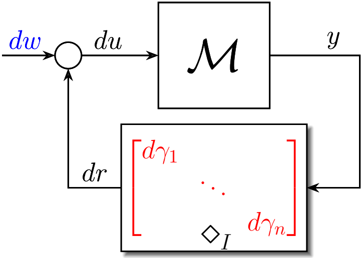
Necessary & Sufficient Conditions of Mean-Square Stability:

- Forward Block is Stable (Finite H^2-norm)
- Spectral Radius of $\mathbb{L}$ is strictly less than 1, $\rho(\mathbb{L}) < 1$

Loop Gain Operator & Mean-Square Stability



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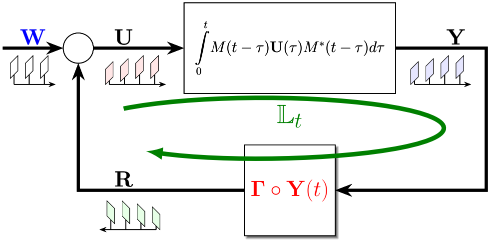

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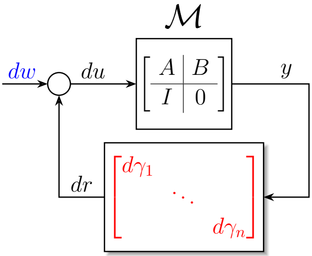
Two important quantities related to $\mathbb{L}$:

- Spectral Radius: $\rho(\mathbb{L})$
- Worst-Case Covariance: $\mathbb{L}(\hat{\mathbf{U}}) = \rho(\mathbb{L})\hat{\mathbf{U}}$ (Perron-Frobenius "Eigen-matrix")

Concluding Remarks & Future Work



SDE: $dy(t) = Ay(t)dt + B\mathrm{Diag}\Big(d\gamma(t)\Big)y(t) + Bdw(t)$

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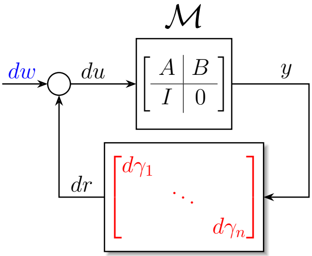

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Extends and unifies the analysis for systems $\mathcal{M}$:

- State space realizations
- Infinite dimensional systems with finite number of multiplicative disturbances
- Systems with delays

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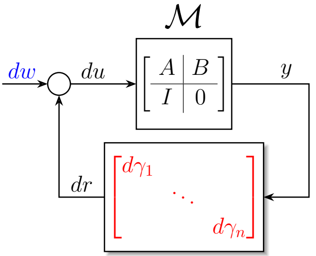

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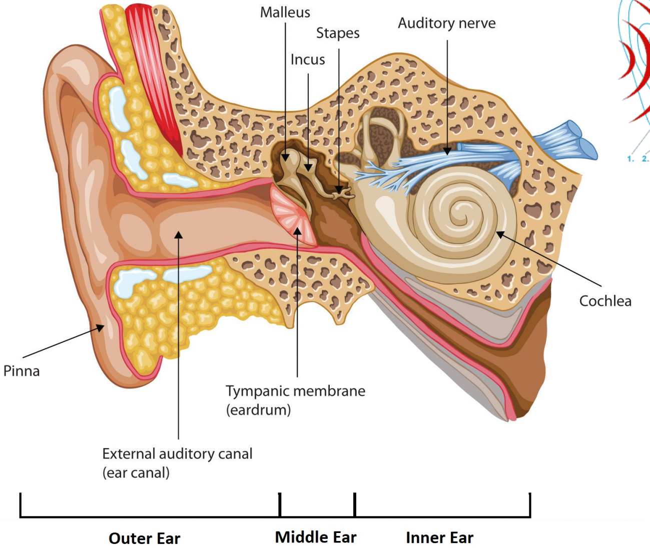
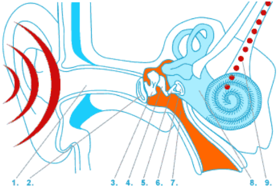
Future Direction: Extend the analysis for

- *Colored* disturbances
- Spatially distributed disturbances with symmetries.

Plan

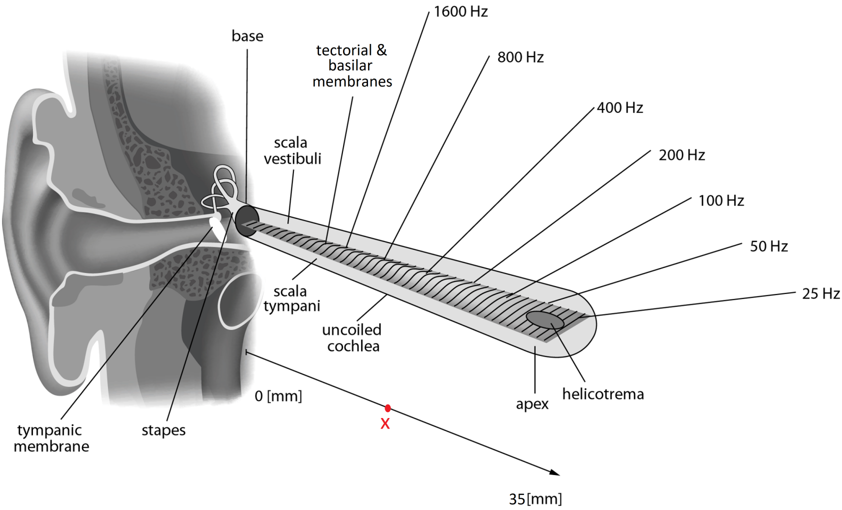
Brief Physiology, the Ear





Source: http://www.byronshvhearing.com/

Brief Physiology, the Cochlea



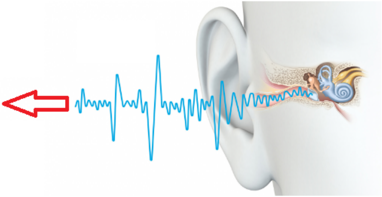
Cochlea is simply a mechanical spectrum analyzer

Source: Biophysical Parameters Modification Could Overcome Essential Hearing Gaps

Spontaneous Response: Cochlear Instabilities

The ear is an active device that can produce sound!

- **Spontaneous Otoacoustic Emissions (SOAE)**
  (Not necessarily perceived)



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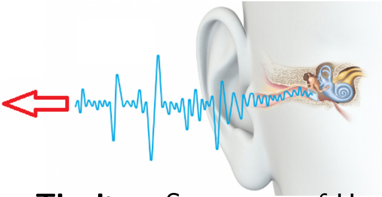

- **Tinnitus**: Symptoms of Hearing Loss Diseases
  (Perceived as harsh and consistent ringing)

Spontaneous Response: Cochlear Instabilities

$\longrightarrow$ Can be modeled as instabilities in stochastic cochlear dynamics...

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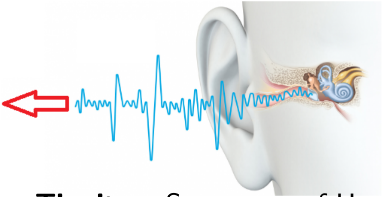

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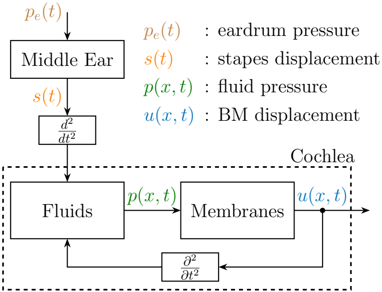
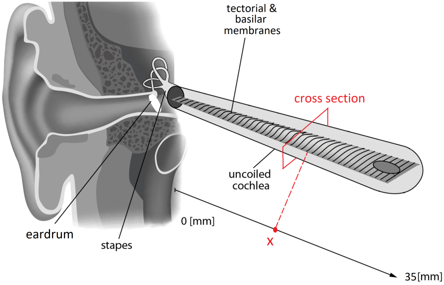
Biomechanical Model



$$p(x,t) = -[\mathcal{M}_f \ddot{u}](x,t) - [\mathcal{M}_s \ddot{s}](x,t)$$

where:

$\mathcal{M}_f$ and $\mathcal{M}_s$ are linear spatial operators



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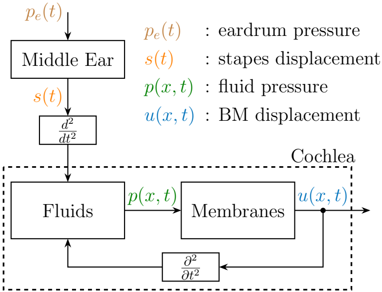
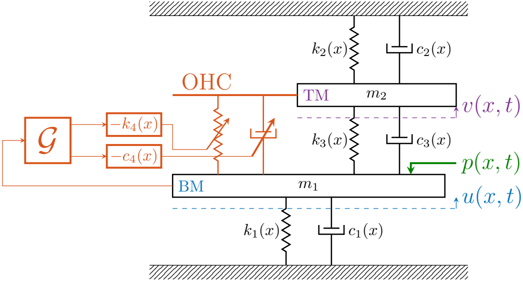
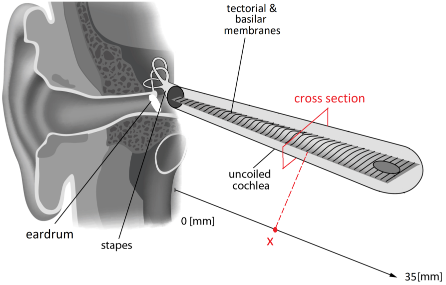


$p_e(t)$: eardrum pressure
$s(t)$: stapes displacement
$p(x,t)$: fluid pressure
$u(x,t)$: BM displacement



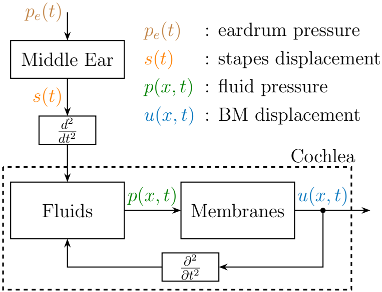
$v(x,t)$: TM displacement
$\gamma(x)$: gain coefficient

$$[\mathcal{G}(u)]\,(x,t) = \frac{\gamma(x)}{1+\theta[\Phi_\eta(u^2)](x,t)}$$



$\mathcal{G}$: active gain mechanism:
small u $\rightarrow$ higher negative damping
$\rightarrow$ large gain
(gives wide dynamic range!)

Biomechanical Model (with stochastic uncertainty)



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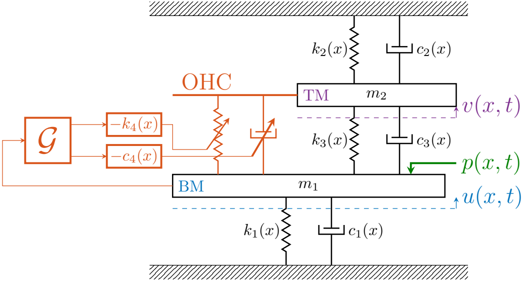

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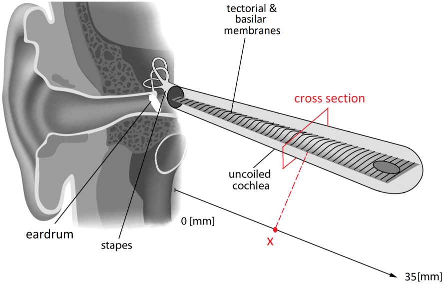


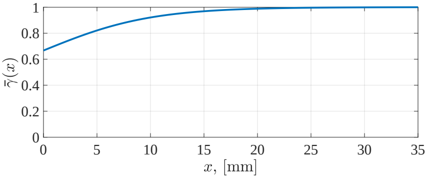
$$[\mathcal{G}(u)](x,t) = \frac{\bar{\gamma}(x)+\tilde{\gamma}(x,t)}{1+\theta[\Phi_\eta(u^2)](x,t)}$$

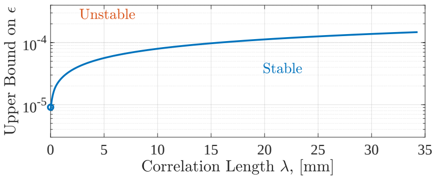
$\tilde{\gamma}(x,t)$: Random Field

MSS Analysis of the Cochlea

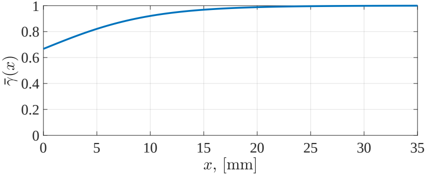
$\gamma(x,t) = \bar{\gamma}(x) + \tilde{\gamma}(x,t)$

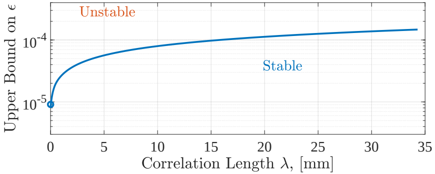
Covariance: $\mathbb{E}\left[\tilde{\gamma}(x,t)\tilde{\gamma}(\xi,\tau)\right] = \dfrac{\epsilon^2}{\lambda\sqrt{2\pi}} e^{\frac{(x-\xi)^2}{2\lambda^2}} \delta(t-\tau)$

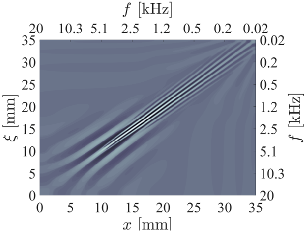




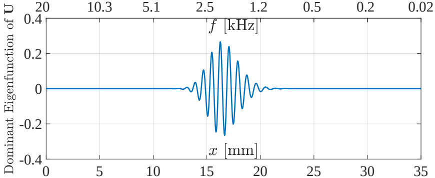
$$\gamma(x,t) = \bar{\gamma}(x) + \tilde{\gamma}(x,t) \qquad \text{Covariance: } \mathbb{E}\left[\tilde{\gamma}(x,t)\tilde{\gamma}(\xi,\tau)\right] = \frac{\epsilon^2}{\lambda\sqrt{2\pi}} e^{\frac{(x-\xi)^2}{2\lambda^2}} \delta(t-\tau)$$





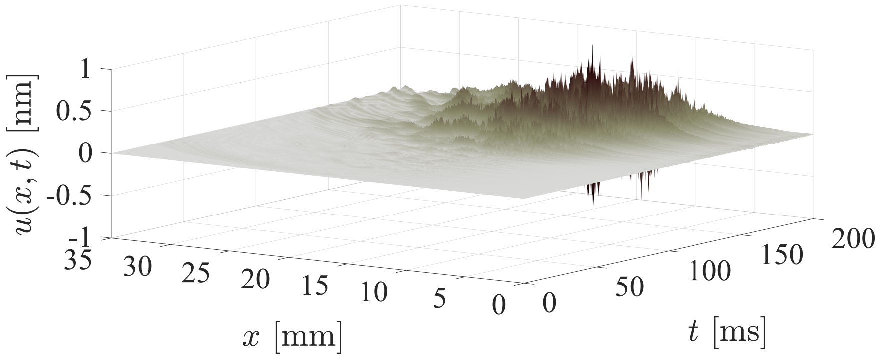


$\mathbf{U}(x,\xi)$: worst case covariance of membrane displacement

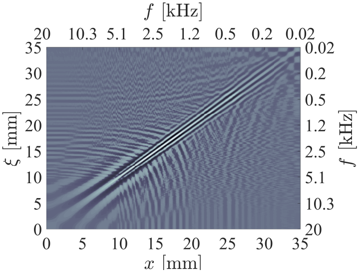


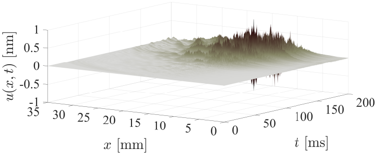
1^{st} eigenfunction of $\mathbf{U}(x,\xi)$ (1^{st} term in K-L expansion)

Stochastic Simulation of the Nonlinear Cochlear Dynamics



$u(x,t)$: Basilar membrane displacement at location x and at time t.





$\longleftarrow$ Empirical Covariance

$$\mathbf{U}_{\mathsf{emp}} \approx \frac{1}{t_f}\int_0^{t_f} u(x,\tau)u(\xi,\tau)d\tau$$

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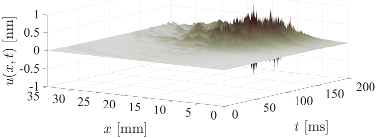

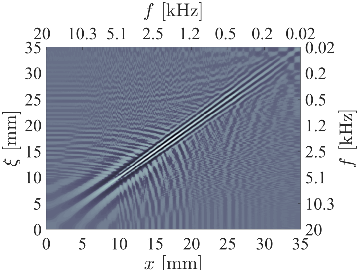

⟵ Empirical Covariance

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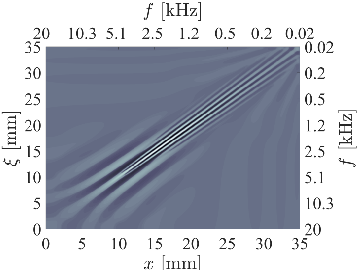
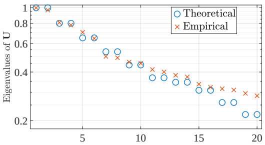
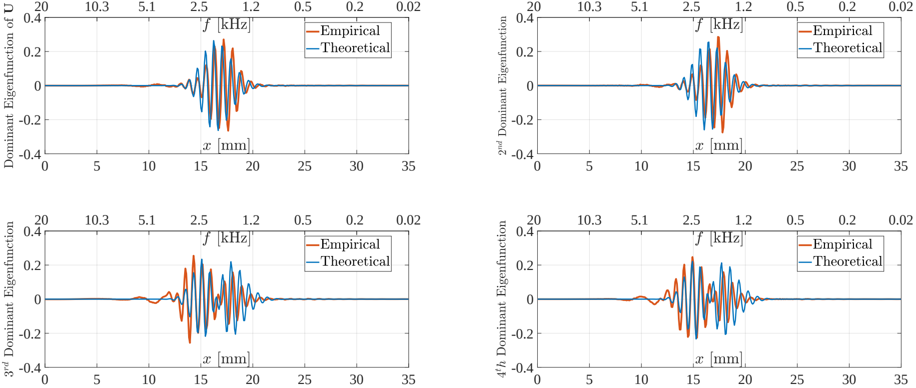


⟵ Predicted Worst-Case Covariance

Simulation-free analysis using the loop gain operator.

Stochastic Simulation of the Nonlinear Cochlear Dynamics





No significant difference: Nonlinearity only saturates the unstable response!

Implications for Control

- Cochlear Models are extremely sensitive to stochastic uncertainties
- Linearized MSS analysis appears predictive of the instabilities
- Can it be used for "control design"?
    - e.g. design input signal with minimal volume to suppress instabilities?

Plan

Problem Formulation: Function Space Approach

$$\begin{aligned} &\underset{x,u}{\text{minimize}} \quad J(x,u) = \frac{1}{2}\int_0^T x^*(t)Qx(t) + u^*(t)Ru(t) \quad dt \\ &\text{subject to} \quad \dot{x}(t) = f\big(x(t), u(t)\big); \quad x(0) = x_0 \end{aligned}$$

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- First Order Method: Gradient Descent $\longrightarrow$ Cheap but Slow Convergence
- Second Order Method: Newton $\longrightarrow$ Fast Convergence but Expensive

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Proposed Method: Keep cost functional & Dynamics separate!

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Dynamical Constraint Set (Trajectories Manifold):

$$x = \mathcal{H}(u) \iff z \in \mathcal{M} \qquad \mathcal{M} = \Big\{ z = (x,u) : x = \mathcal{H}(u) \Big\}$$

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Precondition Constrained-Gradient Descent (PCGD)

$$\begin{array}{ll} \underset{z}{\text{minimize}} & J(z) = \frac{1}{2}\langle z, Hz\rangle \\ \text{subject to} & z \in \mathcal{M} \end{array}$$

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Two Key ideas:

$\longrightarrow$ Two different types of projections
$\longrightarrow$ Preconditioning the state-control space (z-space)

Key Idea 1: Projections...

$$\begin{aligned} &\underset{z}{\text{minimize}} && J(z) = \frac{1}{2}\langle z, z\rangle \qquad (H = I) \\ &\text{subject to} && z \in \mathcal{M} \end{aligned}$$

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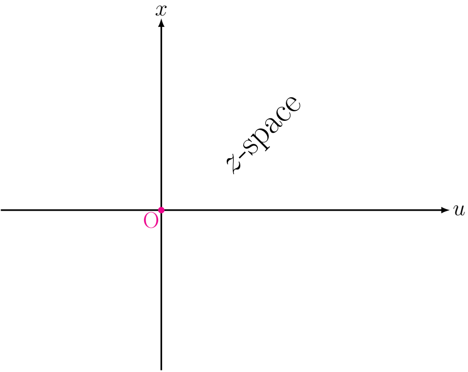

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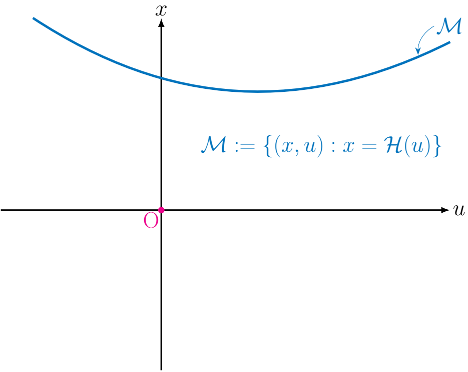

- $\mathcal{M}$: Dynamical Constraints Manifold

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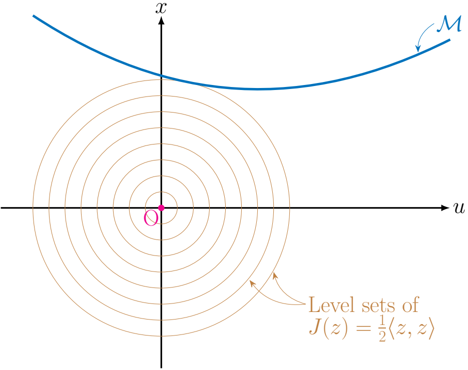

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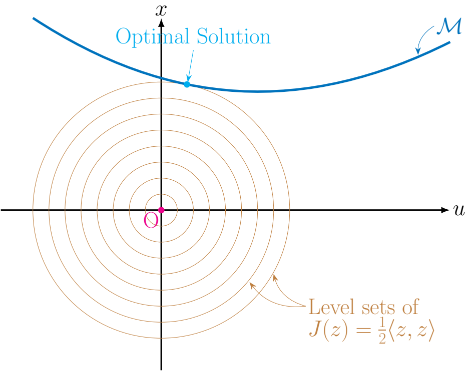

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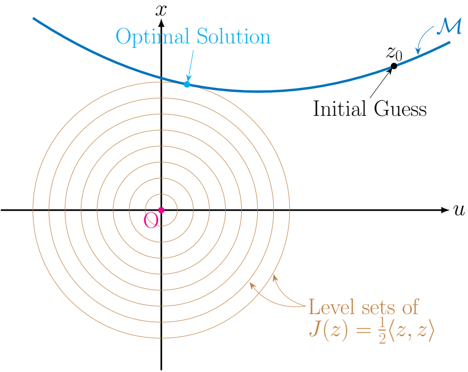

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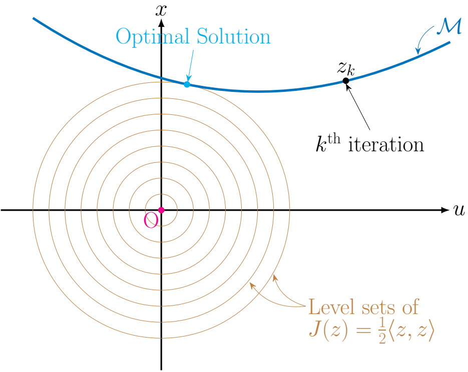

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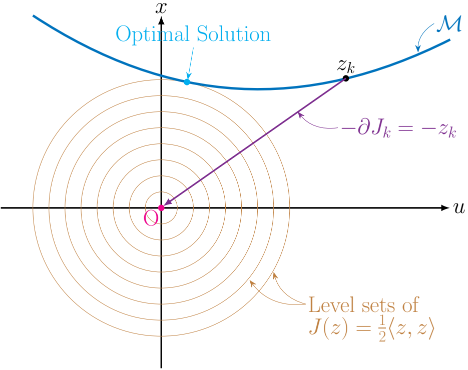

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- ∂J_k: Gradient at Current Iteration

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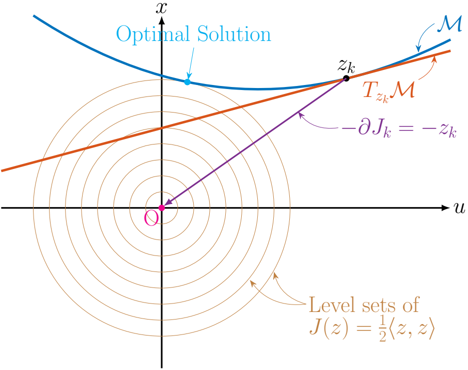

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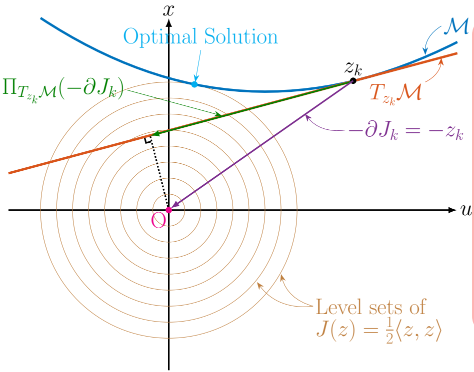

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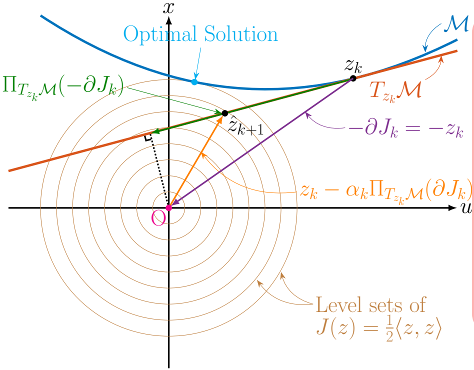

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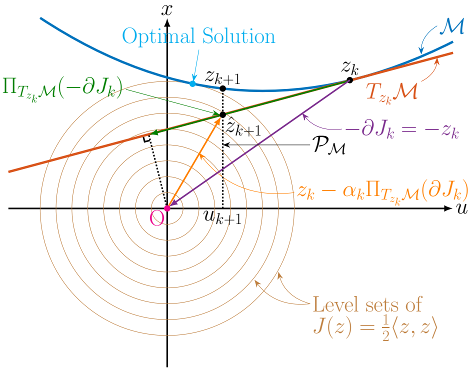

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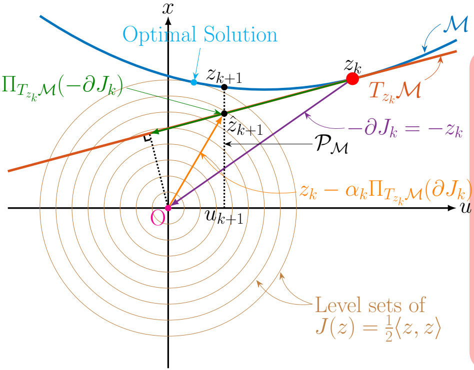

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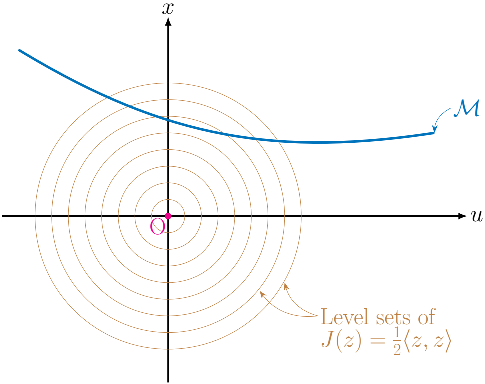
For Spherical Level Sets:
$$\begin{cases} \hat{z}_{k+1} = z_k - \alpha_k \Pi_{T_{z_k}\mathcal{M}}(\partial J_k) \\ z_{k+1} = \mathcal{P}_\mathcal{M}\left(\hat{z}_{k+1}\right) \end{cases}$$

Special Case: Linear Dynamics

$$\underset{z}{\text{minimize}} \quad J(z) = \frac{1}{2}\langle z, z\rangle$$

$$\text{subject to} \quad z \in \mathcal{M} \qquad \mathcal{M} := \{z = (x,u) : \dot{x} = Ax + Bu;\ x(0) = x_0\}$$



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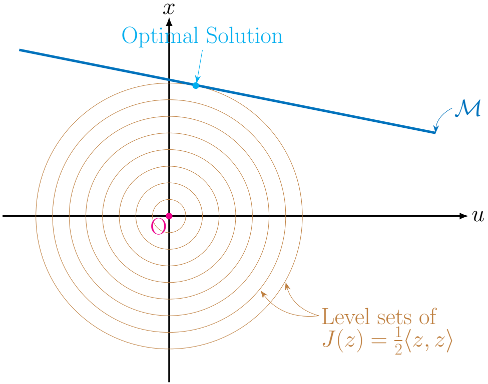

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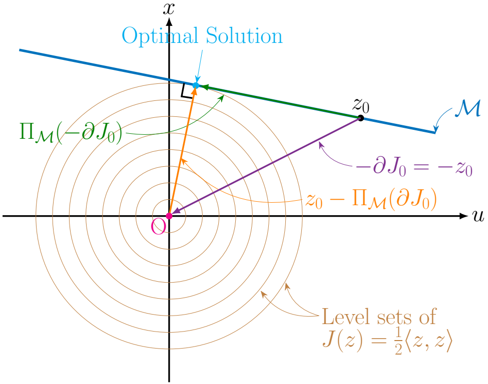

Converges in one iteration with step size $\alpha = 1$!

Special Case: Linear Dynamics

$$\underset{z}{\text{minimize}} \quad J(z) = \frac{1}{2}\langle z, Hz\rangle \qquad (H \neq I)$$

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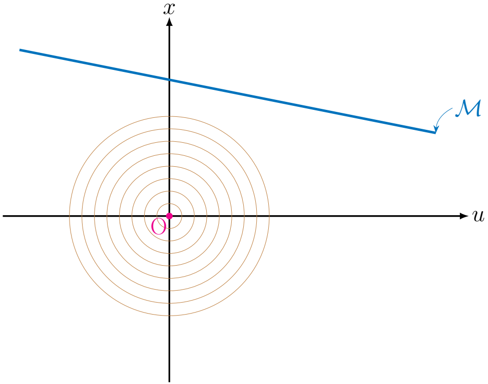

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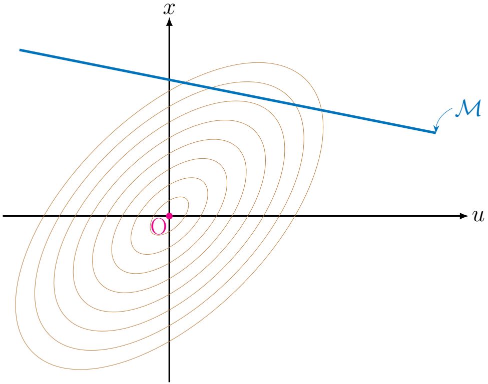

Ellipsoidal level sets: does not converge in one iteration!

Key Idea 2: Preconditioning...

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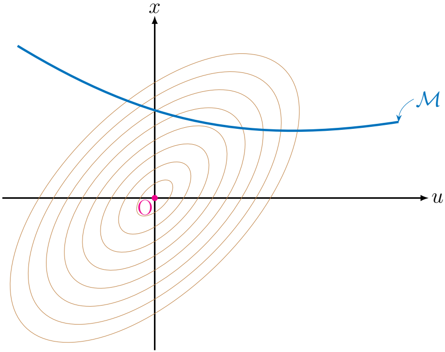

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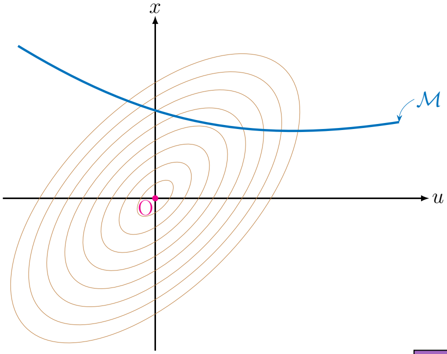
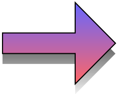


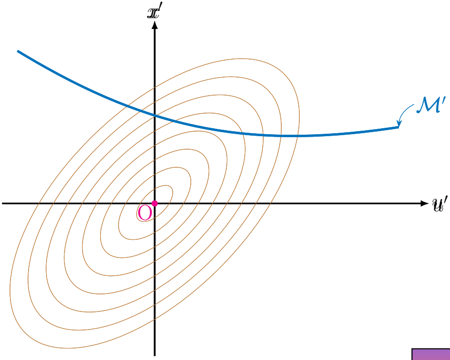


Linear Transformation W: $z' = W(z)$

Key Idea 2: Preconditioning...

$$\underset{z}{\text{minimize}} \quad J(z) = \frac{1}{2}\langle z, Hz\rangle$$
$$\text{subject to} \quad z \in \mathcal{M}$$

$$\underset{z'}{\text{minimize}} \quad J'(z') = \frac{1}{2}\langle z', z'\rangle$$
$$\text{subject to} \quad z' \in \mathcal{M}'$$



Transformation W: $z' = W(z)$

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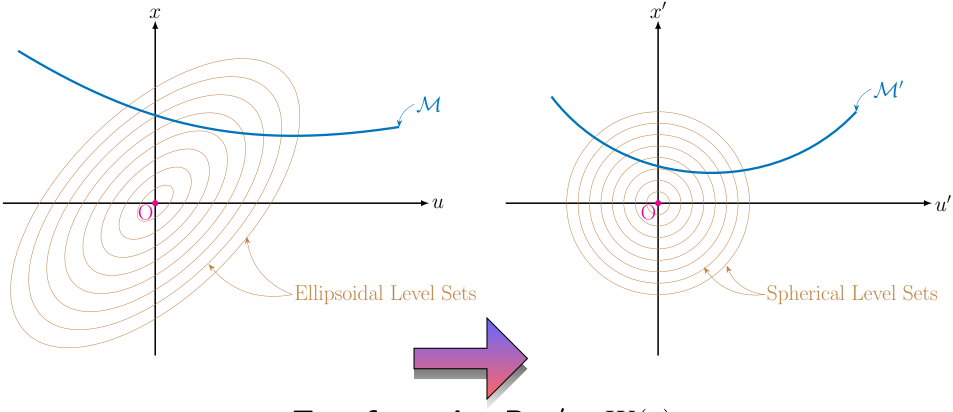


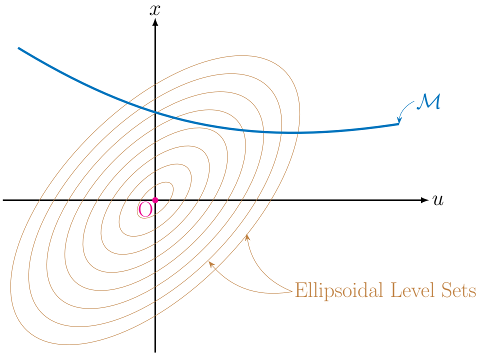
Transformation P: $z' = W(z)$

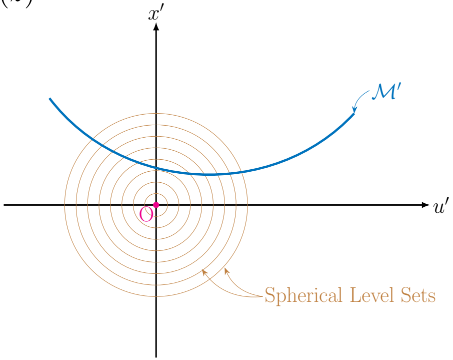
Key Idea 2: Preconditioning...

$$\underset{z}{\text{minimize}} \quad J(z) = \frac{1}{2}\langle z, Hz\rangle \qquad \text{subject to} \quad z \in \mathcal{M}$$

$$z' = W(z)$$

$$\underset{z'}{\text{minimize}} \quad J'(z') = \frac{1}{2}\langle z', z'\rangle \qquad \text{subject to} \quad z' \in \mathcal{M}'$$





$$\begin{cases} \hat{z}'_{k+1} = z'_k - \alpha_k \Pi_{T_{z'_k}\mathcal{M}'}\left(\partial J'_k\right) \\ z_{k+1} = \mathcal{P}_{\mathcal{M}} \circ W^{-1}\left(\hat{z}'_{k+1}\right) \end{cases}$$

Key Idea 2: Preconditioning...

$$\underset{z}{\text{minimize}} \quad J(z) = \frac{1}{2}\langle z, Hz\rangle$$
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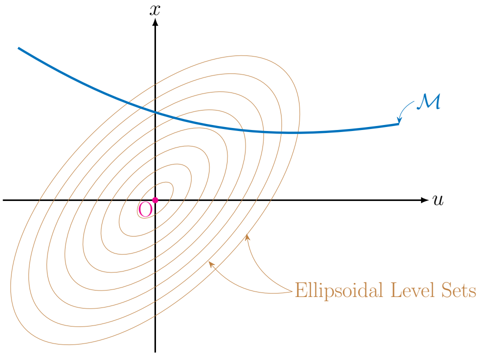

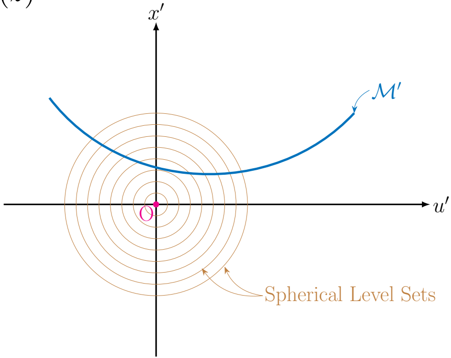

$$\begin{cases} \hat{z}_{k+1} = z_k - \alpha_k \Pi^{H}_{T_{z_k}\mathcal{M}}\left(H^{-1}\partial J_k\right) \\ z_{k+1} = \mathcal{P}_{\mathcal{M}}(\hat{z}_{k+1}) \end{cases}$$

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$$\underset{z}{\text{minimize}} \quad J(z) = \frac{1}{2}\langle z, Hz\rangle$$
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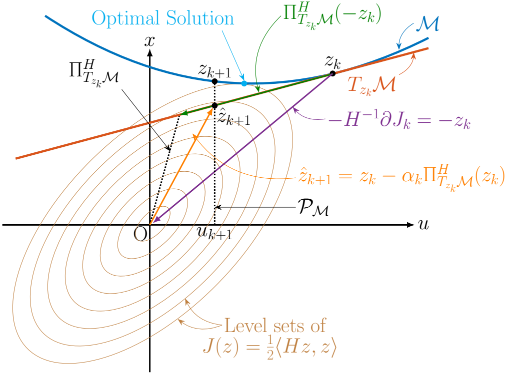

$$\begin{cases} \hat{z}_{k+1} = z_k - \alpha_k \Pi^{\color{red}H}_{T_{z_k}\mathcal{M}}\left({\color{red}H^{-1}}\partial J_k\right) \\ z_{k+1} = \mathcal{P}_\mathcal{M}(\hat{z}_{k+1}) \end{cases}$$

Computational Load

$$\begin{cases} \hat{z}_{k+1} = z_k - \alpha_k \Pi^{H}_{T_{z_k}\mathcal{M}} \left(H^{-1} \partial J_k \right) \\ z_{k+1} = \mathcal{P}_{\mathcal{M}}(\hat{z}_{k+1}) \end{cases}$$

- $\Pi^{H}_{T_{z_k}\mathcal{M}}$: Solve a linear two point boundary value problem

Computational Load

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- $\Pi^{H}_{T_{z_k}\mathcal{M}}$: Solve a linear two point boundary value problem
- $\mathcal{P}_{\mathcal{M}}(\hat{z}_{k+1})$: Solve the system dynamics

Computational Load

$$\begin{cases} \hat{z}_{k+1} = z_k - \alpha_k \Pi^{H}_{T_{z_k}\mathcal{M}} \left(H^{-1} \partial J_k \right) \\ z_{k+1} = \mathcal{P}_{\mathcal{M}}(\hat{z}_{k+1}) \end{cases}$$

- $\Pi^{H}_{T_{z_k}\mathcal{M}}$: Solve a linear two point boundary value problem
- $\mathcal{P}_{\mathcal{M}}(\hat{z}_{k+1})$: Solve the system dynamics
- No Costate Equation!

Computational Load

$$
\begin{cases}
\hat{z}_{k+1} = z_k - \alpha_k \Pi^{H}_{T_{z_k}\mathcal{M}} \left(H^{-1} \partial J_k \right) \\
z_{k+1} = \mathcal{P}_{\mathcal{M}}(\hat{z}_{k+1})
\end{cases}
$$

- $\Pi^{H}_{T_{z_k}\mathcal{M}}$: Solve a linear two point boundary value problem
- $\mathcal{P}_{\mathcal{M}}(\hat{z}_{k+1})$: Solve the system dynamics
- No Costate Equation!
- No second derivatives of the dynamics!

Example: Comparison with the Standard Gradient Descent

$$\underset{x,u}{\text{minimize}} \quad J(x,u) = \frac{1}{2}\int_0^T \left[|\psi(t)^* Q \psi(t)| + Ru^2(t)\right] dt$$

$$\text{subject to} \quad i\hbar \frac{d}{dt}\psi(t) = [H_0 + Vu(t)]\psi(t); \quad \psi(0) = \psi_0$$

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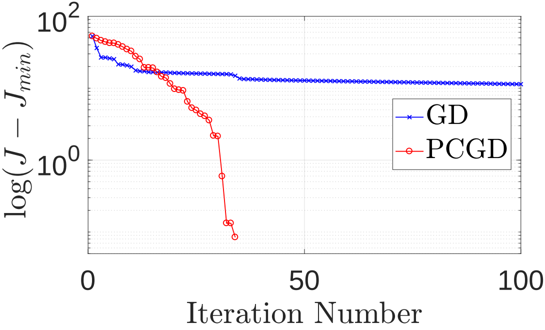
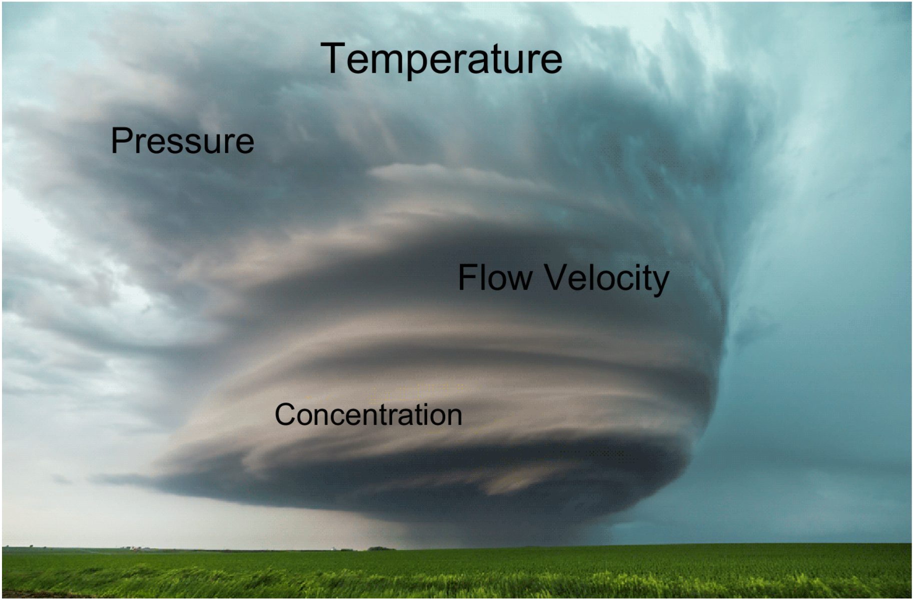
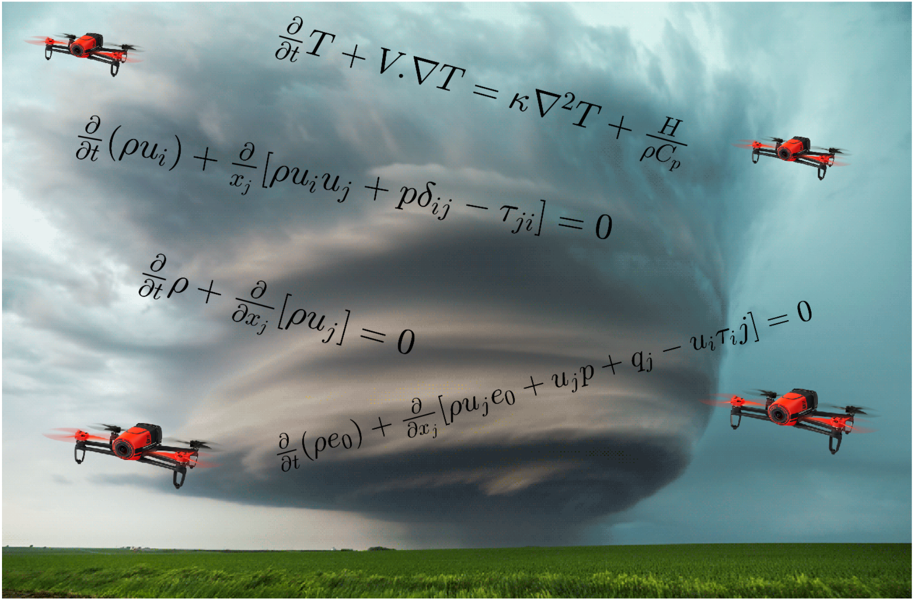


Plan

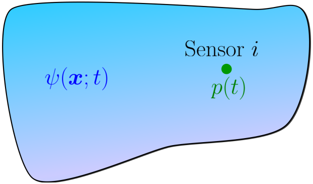


Source: gifsboom.net

Dynamic Estimation: Incorporate the physical laws in the estimation process to reduce the number of sensors needed.



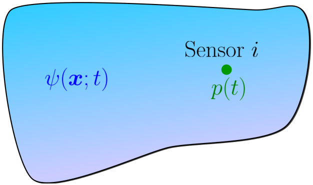
Pointwise Measurement Scheme



$\psi(\mathbf{x}, t)$: unknown field to be estimated in space $\mathbf{x}$ and time t

$p(t)$: sensor position

Pointwise Measurement Scheme



$\psi(\mathbf{x}, t)$: unknown field to be estimated in space $\mathbf{x}$ and time t

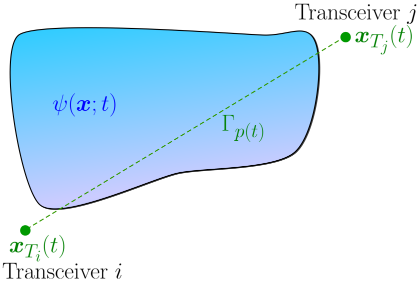
$p(t)$: sensor position

Measurement Equation: $m(t) = \mathcal{C}_{p(t)}\psi(t)$

$$\mathcal{C}_{p(t)}\psi := \psi(p(t);t)$$

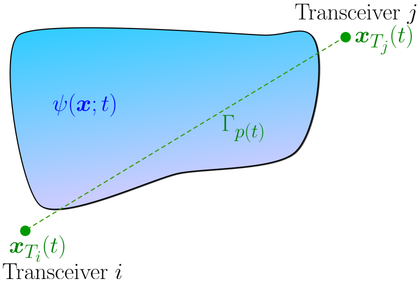
Pointwise Evaluation Operator

Tomographic Measurement Scheme: Line Integrals



$\Gamma_{p(t)}$: time varying line parametrized by $p(t)$

Tomographic Measurement Scheme: Line Integrals

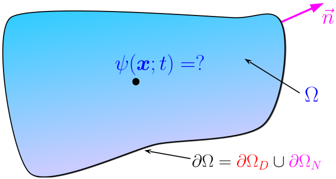


$\Gamma_{p(t)}$: time varying line parametrized by $p(t)$

Measurement Equation: $m(t) = \mathcal{C}_{p(t)}\psi(t)$

$$\mathcal{C}_{p(t)}\psi := \int_{\Gamma_{p(t)}} \psi(\boldsymbol{x};t)d\boldsymbol{x}$$

Line Integral Operator

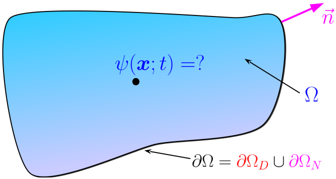


$$\mathbb{E}\left[w(\boldsymbol{x},t)w^*(\boldsymbol{\xi},\tau)\right] = \mathcal{P}(\boldsymbol{x},\boldsymbol{\xi})\delta(t-\tau)$$

Spatial Correlation of PDE Model Uncertainty

Dynamics: $\frac{\partial}{\partial t}\psi(t) = \mathcal{A}\psi(t) + w(t);$ $\psi(0) = \psi_0$

Unknown Boundary Conditions as "Process Noise"



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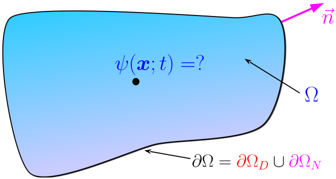
Dynamics: $\frac{\partial}{\partial t}\psi(t) = \mathcal{A}\psi(t) + w(t); \qquad \psi(0) = \psi_0$

BC: $\psi(t)\Big|_{\partial\Omega_D} = \psi_D(t) \qquad \frac{\partial}{\partial \vec{n}}\psi(t)\Big|_{\partial\Omega_N} = \psi_N(t)$

Dirichlet

Neumann

Unknown Boundary Conditions as “Process Noise”



$$\mathbb{E}\left[w(\boldsymbol{x}, t) w^*(\boldsymbol{\xi}, \tau)\right] = \mathcal{P}(\boldsymbol{x}, \boldsymbol{\xi}) \delta(t - \tau)$$

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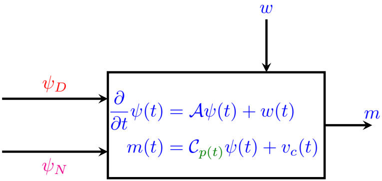
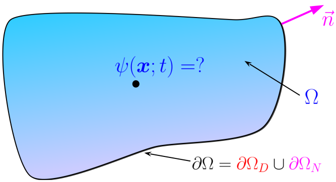


Unknown Boundary Conditions as “Process Noise”



$$\mathbb{E}\left[w(\boldsymbol{x},t)w^*(\boldsymbol{\xi},\tau)\right] = \mathcal{P}(\boldsymbol{x},\boldsymbol{\xi})\delta(t-\tau)$$

$$\mathbb{E}\left[w_D(\boldsymbol{x},t)w_D^*(\boldsymbol{\xi},\tau)\right] = \mathcal{P}_D(\boldsymbol{x},\boldsymbol{\xi})\delta(t-\tau)$$

$$\mathbb{E}\left[w_N(\boldsymbol{x},t)w_N^*(\boldsymbol{\xi},\tau)\right] = \mathcal{P}_N(\boldsymbol{x},\boldsymbol{\xi})\delta(t-\tau)$$

Dynamics: $\frac{\partial}{\partial t}\psi(t) = \mathcal{A}\psi(t) + w(t); \qquad \psi(0) = \psi_0$

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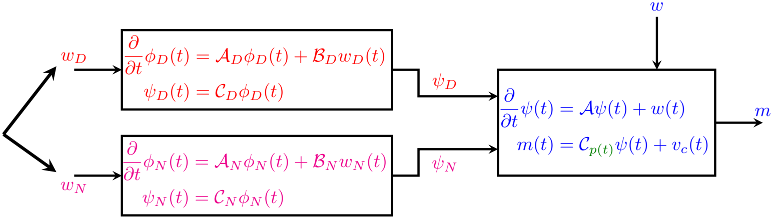
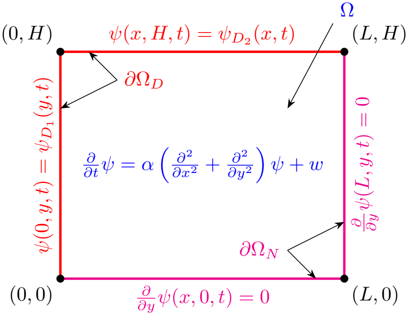
BC Process Noise



Case Study: Dynamic Acoustic Tomography

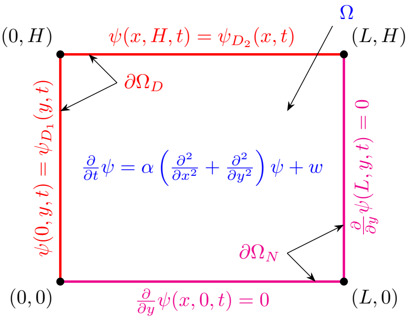


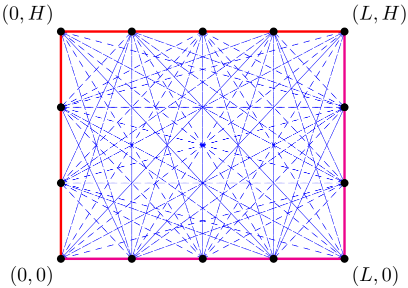
Unknown, Time-Varying Dirichlet BC
(Heated Walls)

Homogeneous Neumann BC
(Insulated Walls)

$$\psi(0, y, t) = 20 + 10\sin\left(\frac{2\pi}{24 \times 60}t\right)$$
$$\psi(x, H, t) = 30 - 10\sin\left(\frac{2\pi}{24 \times 60}t\right)$$

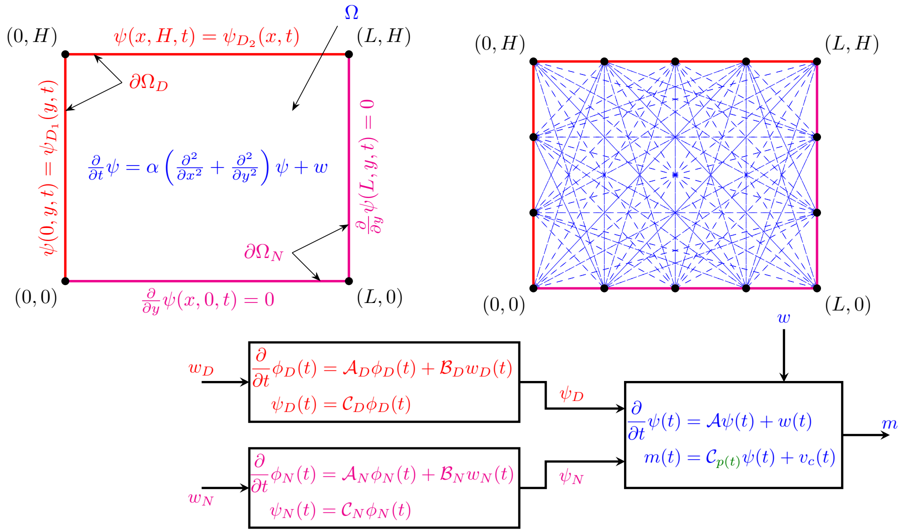
Case Study: Dynamic Acoustic Tomography



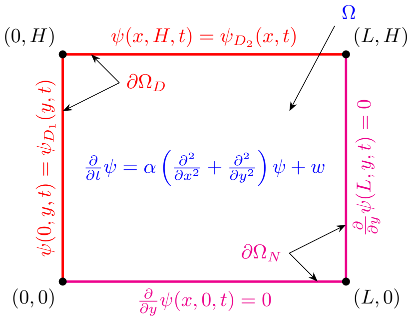


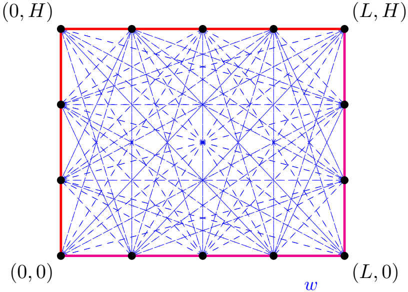
- Ultrasonic transceivers measure the Time of Flight of sound waves.
- Time of Flight depends on the line integral of the temperature field.

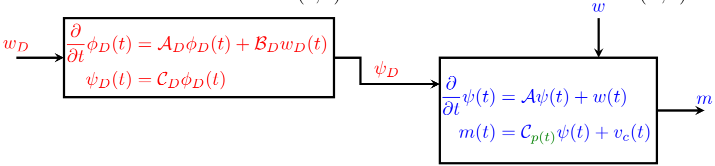
Case Study: Dynamic Acoustic Tomography



Case Study: Dynamic Acoustic Tomography

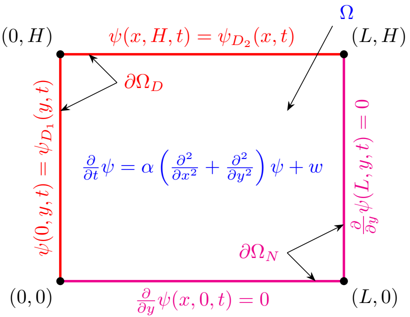






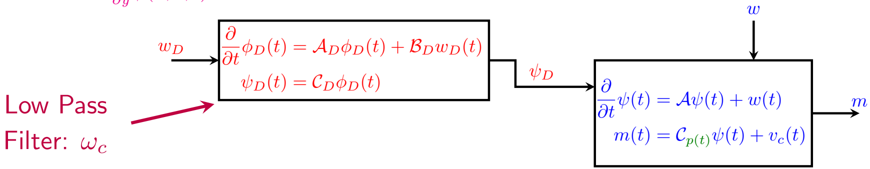
$$\mathbb{E}\left[w_D(\boldsymbol{x},t)w_D^*(\boldsymbol{\xi},\tau)\right]=\mathcal{P}_D(\boldsymbol{x},\boldsymbol{\xi})\delta(t-\tau)$$

Case Study: Dynamic Acoustic Tomography



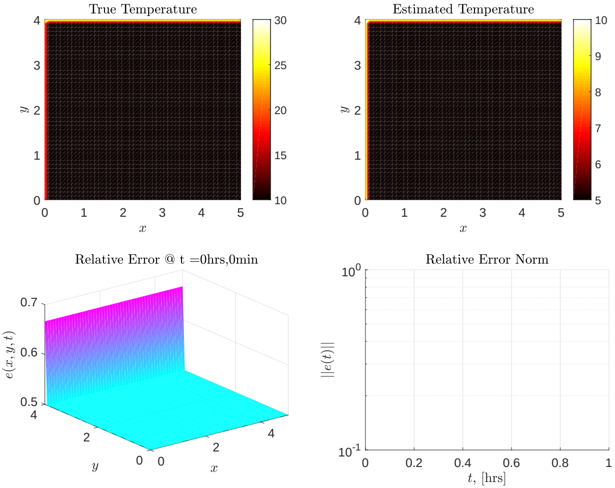
Design Parameters of the Uncertain BC:

- ω_c: Time scale
- a_i: Magnitude
- σ_i: Correlation length



$$\mathcal{P}_D=\begin{bmatrix}\mathcal{P}_{D_1} & 0\\ 0 & \mathcal{P}_{D_2}\end{bmatrix} \qquad \mathcal{P}_{D_i}(x,\xi)=a_ie^{-\frac{(x-\xi)^2}{\sigma_i^2}}; \quad i=1,2$$

Estimation Performance with and without Perfect Knowledge of the Diffusion Constant



Estimation Performance with and without Perfect Knowledge of the Diffusion Constant

Mobile Sensors: Design Objective

$\psi(x,t)$: augmented state space variable
$w(x,t)$: augmented process noise
$v(t)$: measurement noise

Augmented Dynamics:

$$\begin{cases} \dfrac{\partial}{\partial t}\psi(x,t) = \mathcal{A}\psi(\mathbf{x},t) + w(t); \qquad \psi(\mathbf{x},0) = \psi_0(\mathbf{x}) \\ \qquad m(t) = \mathcal{C}_{p(t)}\psi(\mathbf{x},t) + v(t) \end{cases}$$

$\longrightarrow$ **Goal**: Design the path $p(t)$ to minimize the estimation error in some sense.

Optimal Control Problem in Continuous Space-Time

$\hat{\psi}(\boldsymbol{x}, t)$	$\longrightarrow$	**Optimal** State Estimate
$e(\boldsymbol{x}, t) := \psi(\boldsymbol{x}, t) - \hat{\psi}(\boldsymbol{x}, t)$	$\longrightarrow$	Estimation Error
$\mathbb{E}\left[e(\boldsymbol{x}, t) e^*(\boldsymbol{\xi}, \tau)\right] := \mathcal{X}(\boldsymbol{x}, \boldsymbol{\xi}; t)\delta(t - \tau)$	$\longrightarrow$	Estimation Error Covariance

Optimal Control Problem in Continuous Space-Time

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$e(\boldsymbol{x},t) := \psi(\boldsymbol{x},t) - \hat{\psi}(\boldsymbol{x},t)$ $\longrightarrow$ Estimation Error

$\mathbb{E}\left[e(\boldsymbol{x},t)e^*(\boldsymbol{\xi},\tau)\right] := \mathcal{X}(\boldsymbol{x},\boldsymbol{\xi};t)\delta(t-\tau)$ $\longrightarrow$ Estimation Error Covariance

$$\implies \text{trace}(\mathcal{X}(t)) = \mathbb{E}\left[\int e^*(\boldsymbol{\xi},t)e(\boldsymbol{\xi},t)d\boldsymbol{\xi}\right] = \mathbb{E}\left[||e(t)||^2_{L_2}\right]$$

Optimal Control Problem in Continuous Space-Time

$\hat{\psi}(\boldsymbol{x}, t)$ $\longrightarrow$ **Optimal** State Estimate

$e(\boldsymbol{x}, t) := \psi(\boldsymbol{x}, t) - \hat{\psi}(\boldsymbol{x}, t)$ $\longrightarrow$ Estimation Error

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$$\Longrightarrow \text{trace}(\mathcal{X}(t)) = \mathbb{E}\left[\int e^*(\boldsymbol{\xi}, t)e(\boldsymbol{\xi}, t)d\boldsymbol{\xi}\right] = \mathbb{E}\left[||e(t)||_{L_2}^2\right]$$

Objective:

- Design$\{$p(t)$\}$ to minimize tr$(\mathcal{X})$
- Add some penalty on the sensors' mobility

Optimal Control Problem in Continuous Space-Time

$\hat{\psi}(\boldsymbol{x},t) \longrightarrow$ **Optimal** State Estimate

$e(\boldsymbol{x},t) := \psi(\boldsymbol{x},t) - \hat{\psi}(\boldsymbol{x},t) \longrightarrow$ Estimation Error

$\mathbb{E}\left[e(\boldsymbol{x},t)e^*(\boldsymbol{\xi},\tau)\right] := \mathcal{X}(\boldsymbol{x},\boldsymbol{\xi};t)\delta(t-\tau) \longrightarrow$ Estimation Error Covariance

$$\implies \text{trace}(\mathcal{X}(t)) = \mathbb{E}\left[\int e^*(\boldsymbol{\xi},t)e(\boldsymbol{\xi},t)d\boldsymbol{\xi}\right] = \mathbb{E}\left[||e(t)||^2_{L_2}\right]$$

Objective:

- Design$\{$p(t)$\}$ to minimize tr$(\mathcal{X})$
- Add some penalty on the sensors' mobility

$$\min_{\{z(t);\mathcal{X}(t)\}} \int_0^{t_f} \left(tr(\mathcal{X}(t)) + \frac{1}{2}z(t)^T Q_s z(t) + \frac{1}{2}u(t)^T R_s u(t)\right) dt$$

Dynamics of Error Covariance; Sensor Dynamics

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{X} = \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^* + \mathcal{Q} - \mathcal{X}\mathcal{C}_p^* R^{-1}\mathcal{C}_p\mathcal{X}; & \mathcal{X}(0) = \mathcal{X}_0 \\ \frac{d}{dt}z = Fz + Gu; & z(0) = z_0 \\ p = Hz & \end{cases}$$

Deterministic Optimal Control Problem

Necessary Conditions of Optimality: States & Costates

- **Covariance State & Costate:** $\mathcal{X} \longleftrightarrow \mathcal{Y}$

$$\frac{\partial}{\partial t}\mathcal{X} = \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^* + \mathcal{Q} - \mathcal{X}\mathcal{C}_p^* R^{-1}\mathcal{C}_p\mathcal{X}; \qquad \mathcal{X}(0) = 0$$
$$-\frac{\partial}{\partial t}\mathcal{Y} = \left(\mathcal{A} - \mathcal{L}_p\mathcal{C}_p\right)^* \mathcal{Y} + \mathcal{Y}\left(\mathcal{A} - \mathcal{L}_p\mathcal{C}_p\right) + \mathcal{I}; \qquad \mathcal{Y}(t_f) = 0$$

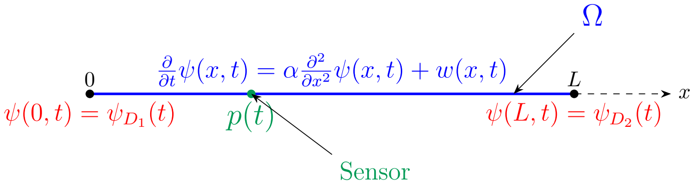
where $\mathcal{L}_p := \mathcal{X}\mathcal{C}_p R^{-1}$ is the Kalman Gain.

- **Sensor State & Costate Equation:** $z \longleftrightarrow \lambda$

$$\frac{d}{dt}z = Fz + Gu; \qquad (u = -R_s^{-1}G^T\lambda); \qquad z(0) = 0$$
$$-\frac{d}{dt}\lambda = F^T\lambda + Q_s z - H^T \mathrm{tr}\left(\mathcal{X}\mathcal{W}_p\mathcal{X}\mathcal{Y}\right); \qquad \lambda(t_f) = 0$$

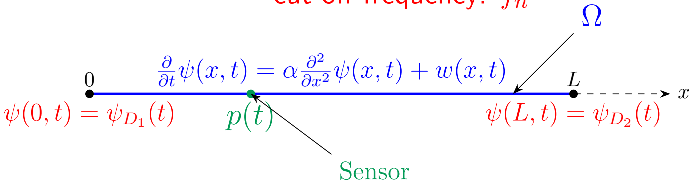
where $\mathcal{W}_p := \frac{\partial}{\partial p}(\mathcal{C}_p^* R^{-1}\mathcal{C}_p)$ and $p = Hz$

Case Study: Sensor Path Design on 1D Heat Equation



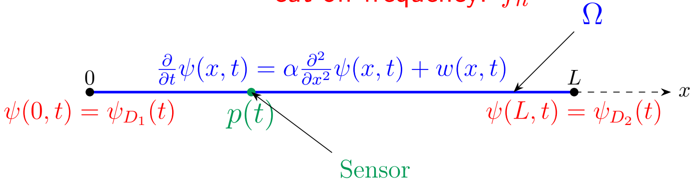
Case Study: Sensor Path Design on 1D Heat Equation

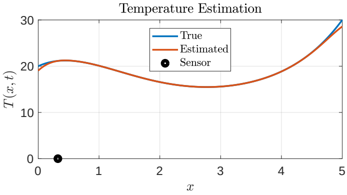
cut off frequency: f_n

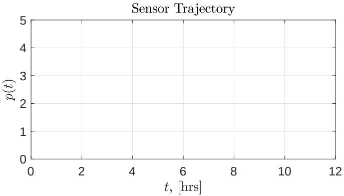


Case Study: Sensor Path Design on 1D Heat Equation

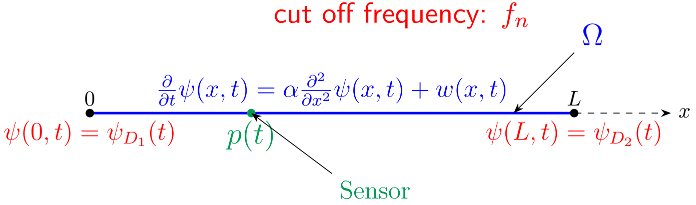
cut off frequency: f_n





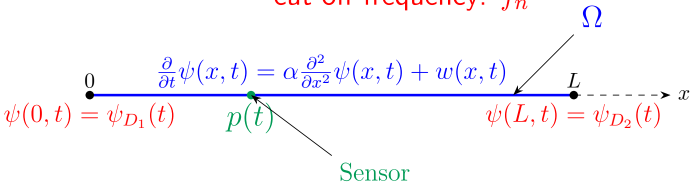


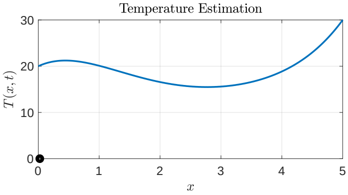
Case Study: Sensor Path Design on 1D Heat Equation

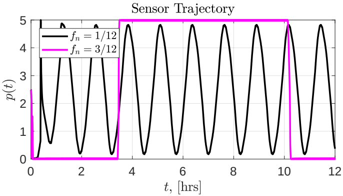


Case Study: Sensor Path Design on 1D Heat Equation

cut off frequency: f_n







Future Work

- Understand the structure of the state/costate differential equations
- Devise efficient numerical methods
- Generalize to nonlinear distributed dynamical systems
- Apply to Navier-Stokes equations

Acknowledgments

Thank you

Publications & Submissions

- Filo and Bamieh, "A Block Diagram Approach to Stochastic Calculus with Application to Multiplicative Uncertainty Analysis", CDC 2018 (submitted).
- Filo and Bamieh, "An Input Output Approach to Structured Stochastic Uncertainty in Continuous Time", Automatic Control, IEEE Transactions on, 2018 (submitted).
- Filo and Bamieh, "Investigating Cochlear Instabilities using Structured Stochastic Uncertainty", CDC 2017.
- Filo and Bamieh, "Stochastic Cochlear Models", Journal of Acoustical Society of America, 2018 (submitted).
- Filo and Bamieh, "Function Space Approach for Gradient Descent in Optimal Control", ACC 2018.
- Filo and Bamieh, "Optimal Control in Function Space." Control Systems Magazine, 2018 (to be submitted).
- Filo and Bamieh, "Sensor Motion for Optimal Estimation in Distributed Dynamic Environments", ACC 2017.
- Bamieh and Filo, "An Input Ouput Approach to Structured Stochastic Uncertainty in Discrete Time", Automatic Control, IEEE Transactions on, 2018 (submitted).